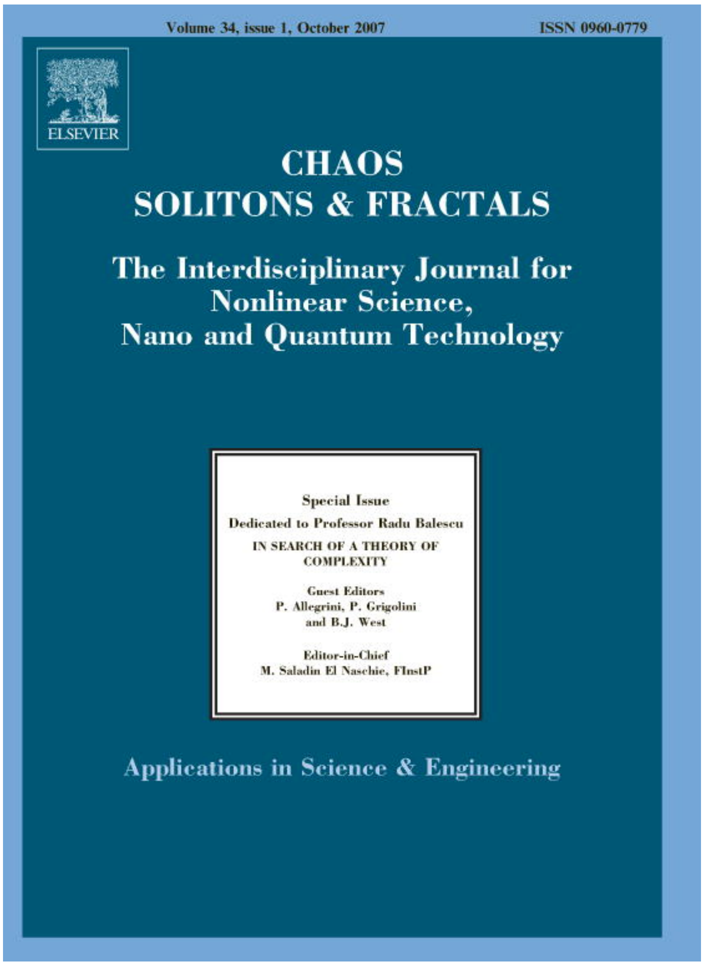
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Some fundamental aspects of Lévy flights

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Abstract

We investigate the physical basis and properties of Lévy flights (LFs), Markovian random walks with a long-tailed density of jump lengths, $\lambda(\xi) \sim |\xi|^{-1-\alpha}$, with $0 < \alpha < 2$. In particular, we show that non-trivial boundary conditions need to be carefully posed, and that the method of images fails due to the non-locality of LFs. We discuss the behaviour of LFs in external potentials, demonstrating the existence of multimodal solutions whose maxima do not coincide with the potential minimum. The Kramers escape of LFs is investigated, and the physical nature of the a priori diverging kinetic energy of an LF is addressed.

1. Introduction

For sums of independent, identically distributed random variables with proper normalization to the sample size, the generalized central limit theorem guarantees the convergence of the associated probability density to a Lévy stable density (LSD) even though the variance of these random variables diverges [1–4]. Well-known examples for LSDs are the one-sided (defined for $x \geqslant 0$) Lévy–Smirnov distribution

$$f_{1/2,-1/2}(x) = \sqrt{\frac{1}{4\pi x^3}} \exp\left(-\frac{1}{4x}\right), \tag{1}$$

related to the first passage time density of a Gaussian random walk process of passing the origin (see below), and the Cauchy (or Lorentz) distribution

$$f_{1,0}(x) = \frac{1}{\pi(1^2 + x^2)}. \tag{2}$$

In general, an LSD is defined through its characteristic function of the probability density $f(x)$

$$\varphi(z) \equiv \mathscr{F}\{f(x)\} = \int_{-\infty}^{\infty} f_{\alpha,\beta}(x) e^{ikx}\, dx \tag{3}$$

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where

$$\log \varphi(z) = -|z|^{\alpha} \exp\left\{ \mathrm{i}\frac{\pi\beta}{2}\,\mathrm{sign}(z) \right\}, \tag{4}$$

for $\alpha \neq 1$. Here, the skewness (or asymmetry) parameter β is restricted to the following region:

$$|\beta| \leqslant \begin{cases} \alpha, & \text{if } 0 < \alpha < 1 \\ 2-\alpha, & \text{if } 1 < \alpha < 2. \end{cases} \tag{5}$$

For $\beta = 0$, the corresponding LSD is symmetric around $x = 0$, while for $\beta = -\alpha$ and $0 < \alpha < 1$, it is one-sided. In general, an LSD follows the power-law asymptotic behaviour

$$f_{\alpha,\beta}(x) \sim \frac{A_{\alpha,\beta}}{|x|^{1+\alpha}}, \quad \alpha < 2, \tag{6}$$

with $A_{\alpha,\beta}$ being a constant, such that for all LSDs with $\alpha < 2$ the variance diverges

$$\langle x^2 \rangle = \infty. \tag{7}$$

Conversely, all fractional moments $\langle |x|^{\delta} \rangle < \infty$ for all $0 < \delta < \alpha \leqslant 2$. From above definitions it is obvious that the LSD $f_{2,0}$ corresponds to the Gaussian normal distribution

$$f_{2,0}(x) = \sqrt{\frac{1}{4\pi}} \exp\left(-\frac{1}{4}x^2 \right) \tag{8}$$

possessing finite moments of any order. In this limit, the generalized central limit theorem coincides with the more traditional central limit theorem.

Random processes whose spatial coordinate x or clock time t are distributed according to an LSD exhibit anomalies, that is, no longer follow the laws of Brownian motion. Consider a continuous time random walk process defined in terms of the jump length and waiting time distributions $\lambda(\xi)$ and $\psi(\tau)$ [5]. Each jump event of this random walk, that is, is characterized by a jump length ξ drawn from the distribution λ, and the time τ between two jump events is distributed according to ψ. (Note that an individual jump is supposed to occur spontaneously.) In absence of an external bias, continuous time random walk theory connects $\lambda(\xi)$ and $\psi(\tau)$ with the probability distribution $P(x,t)\mathrm{d}x$ to find the random walker at a position in the interval $(x, x+\mathrm{d}x)$ at time t. In Fourier–Laplace space, $P(k,u) \equiv \mathscr{F}\{\mathscr{L}\{P(x,t); t \to u\}; x \to k\}$, this relation reads [6]

$$P(k,u) = \frac{1-\psi(u)}{u} \frac{1}{1-\lambda(k)\psi(u)}, \tag{9}$$

where $\mathscr{L}\{f(t)\} \equiv \int_0^{\infty} \exp(-ut) f(t) \mathrm{d}t$. We here neglect potential complications due to ageing effects. The following cases can be distinguished:

(i) $\lambda(\xi)$ is Gaussian with variance σ^2 and $\psi(\tau) = \delta(\tau - \tau_0)$. Then, to leading order in k^2 and u, respectively, one obtains $\lambda(k) \simeq 1 - \sigma^2 k^2$ and $\psi(u) \simeq 1 - u\tau_0$. From relation (9) one recovers the Gaussian probability density $P(x,t) = \sqrt{1/(4\pi K t)} \exp\{-x^2/(4Kt)\}$ with diffusion constant $K = \sigma^2/\tau_0$. The corresponding mean squared displacement grows linearly with time:

$$\langle x^2(t) \rangle = 2Kt. \tag{10}$$

This case corresponds to the continuum limit of regular Brownian motion. Note that here and in the following, we restrict the discussion to one dimension.

(ii) Assume $\lambda(\xi)$ still to be Gaussian, while for the waiting time distribution $\psi(\tau)$ we choose a one-sided LSD with stable index $0 < \alpha < 1$. Consequently, $\psi(u) \simeq 1 - (u\tau_0)^{\alpha}$, and the characteristic waiting time $\int_0^{\infty} \psi(\tau)\tau\,\mathrm{d}\tau$ diverges. Due to this lack of a time scale separating microscopic (single jump events) and macroscopic (on the level of $P(x,t)$) scales, $P(x,t)$ is no more Gaussian, but given by a more complex H-function [7–9]. In Fourier space, however, one finds the quite simple analytical form [8]

$$P(k,t) = E_{\alpha}(-K_{\alpha}k^2 t^{\alpha}) = \sum_0^{\infty} \frac{(K_{\alpha}k^2 t^{\alpha})^n}{\Gamma(1+\alpha n)} \tag{11}$$

in terms of the Mittag–Leffler function. This generalized relaxation function of the Fourier modes turns from an initial stretched exponential (KWW) behaviour $P(k,t) \sim 1 - K_\alpha k^2 t^\alpha/\Gamma(1+\alpha) \sim \exp\{-K_\alpha k^2 t^\alpha/\Gamma(1+\alpha)\}$ to a terminal power-law behaviour $P(k,t) \sim (K_\alpha k^2 t^\alpha \Gamma(1-\alpha))^{-1}$ [8]. In the limit $\alpha \to 1$, it reduces to the traditional exponential $P(k,t) = \exp(-Kk^2t)$ with finite characteristic relaxation time. Also the mean squared displacement changes from its linear to the power-law time dependence

$$\langle x^2(t)\rangle = 2K_\alpha t^\alpha, \tag{12}$$

with $K_\alpha = \sigma^2/\tau_0^\alpha$. This is the case of *subdiffusion*. We note that in x,t space the dynamical equation is the fractional diffusion equation [7]. In the presence of an external potential, it generalizes to the time-fractional Fokker–Planck equation [10,8,9].

(iii) Finally, take $\psi(\tau) = \delta(\tau - \tau_0)$ sharply peaked, but $\lambda(\xi)$ of Lévy stable form with index $0 < \alpha < 2$. The resulting process is Markovian, but with diverging variance. It can be shown that the fractional moments scale like [11]

$$\langle |x(t)|^\delta\rangle \propto (K_\alpha t)^{\delta/\alpha}, \tag{13}$$

were $K_\alpha = \sigma^\alpha/\tau_0$. From Eq. (9) one can immediately obtain the Fourier image of the associated probability density function,

$$P(k,t) = \exp\{-K_\alpha |k|^\alpha t\}. \tag{14}$$

From Eq. (4) this is but a symmetric LSD with stable index α, and this type of random walk process is most aptly coined a Lévy flight. A Lévy flight manifestly has regular exponential mode relaxation and is in fact Markovian (see below). However, the modes in position space are no more sharply localized like in the Gaussian or subdiffusive case. Instead, individual modes bear the hallmark of an LSD, that is, the diverging variance. We will see below how the presence of steeper than harmonic external potentials cause a finite variance of the Lévy flight, although a power-law form of the probability density remains.

In the remainder of this paper, we deal with the physical and mathematical properties of Lévy flights in the presence of external force fields. While mostly we will be concerned with the overdamped case, in the last section we will address the dynamics in velocity space in the presence of Lévy noise, in particular, the question of the diverging variance of Lévy flights.

2. Underlying random walk process

To derive the dynamic equation of a Lévy flight in the presence of an external force field $F(x) = -\mathrm{d}V(x)/\mathrm{d}x$, we pursue two different routes. One starts with a generalized version of the continuous time random walk, compare Ref. [12] for details.

To include the local asymmetry of the jump length distribution due to the force field $F(x)$, we introduce [12,13] the generalized transfer kernel $\Lambda(x,x') = \lambda(x-x')[A(x')\Theta(x-x') + B(x')\Theta(x'-x)]$ (and therefore $\Lambda(x,x') = \Lambda(x';x-x')$). As in standard random walk theory (compare [14]), the coefficients $A(x)$ and $B(x)$ define the local asymmetry for jumping left and right, depending on the value of $F(x)$. Here, $\Theta(x)$ is the Heaviside jump function. With the normalization $\int \Lambda(x',\Delta)\mathrm{d}\Delta = 1$, the fractional Fokker–Planck equation (FFPE) ensues [12]:

$$\frac{\partial}{\partial t}P(x,t) = \left(-\frac{\partial}{\partial x}\frac{F(x)}{m\eta} + K_\alpha \frac{\partial^\alpha}{\partial |x|^\alpha}\right)P(x,t). \tag{15}$$

Remarkably, the presence of the Lévy stable $\lambda(\xi)$ only affects the diffusion term, while the drift term remains unchanged [15,12]. The fractional spatial derivative represents an integrodifferential operator defined through

$$\frac{\partial^\alpha}{\partial |x|^\alpha}P(x,t) = \frac{-1}{2\cos(\pi\alpha/2)\Gamma(2-\alpha)}\frac{\partial^2}{\partial x^2}\int_{-\infty}^{\infty}\frac{P(x',t)}{|x-x'|^{\alpha-1}}, \tag{16}$$

for $1 < \alpha < 2$, and a similar form for $0 < \alpha < 1$ [16–18]. In Fourier space, for all $0 < \alpha \leqslant 2$ the simple relation

$$\mathscr{F}\left\{\frac{\partial^\alpha}{\partial |x|^\alpha}P(x,t)\right\} = -|k|^\alpha P(k,t) \tag{17}$$

holds. In the Gaussian limit $\alpha = 2$, all relations above reduce to the familiar second-order derivatives in x and thus the corresponding $P(x,t)$ is governed by the standard Fokker–Planck equation.

The FFPE (15) can also be derived from the Langevin equation [15,19,16]

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -\frac{1}{m\gamma}\frac{\mathrm{d}V(x)}{\mathrm{d}x} + \xi_\alpha(t), \tag{18}$$

driven by white Lévy stable noise $\xi_\alpha(t)$, defined through $L(\Delta t) = \int_t^{t+\Delta t}\xi_\alpha(t')\mathrm{d}t'$ being a symmetric LSD of index α with characteristic function $p(k,\Delta t) = \exp(-K_\alpha|k|^\alpha\Delta t)$ for $0 < \alpha \leqslant 2$. As with standard Langevin equations, K_α denotes the noise strength, m is the mass of the diffusing (test) particle, and γ is the friction constant characterizing the dissipative interaction with the bath of surrounding particles.

A subtle point about the FFPE (15) is that it does not uniquely define the underlying trajectory [20]; however, starting from our definition of the process in terms of the stable jump length distribution $\lambda(\xi) \sim |x|^{-1-\alpha}$, or its generalized pendant $\Lambda(x,x')$, the FFPE (15) truly represents a Lévy flight in the presence of the force $F(x)$. This poses certain difficulties when non-trivial boundary conditions are involved, as shown below.

3. Propagator and symmetries

In absence of an external force, $F(x) = 0$, the exact solution of the FFPE is readily obtained as the LSD $P(k,t) = \exp(-K_\alpha|k|^\alpha t)$ in Fourier space. Back-transformed to position space, an analytical solution is given in terms of the Fox H-function [8,21,19]

$$P(x,t) = \frac{1}{\alpha|x|}H^{1,1}_{2,2}\left[\frac{|x|}{(K_\alpha t)^{1/\alpha}}\middle|\begin{matrix}(1,1/\alpha),(1,1/2)\\(1,1),(1,1/2)\end{matrix}\right], \tag{19}$$

from which the series expansion

$$P(x,t) = \frac{1}{\alpha(K_\alpha t)^{1/\alpha}}\sum_{\nu=0}^{\infty}\frac{\Gamma([1+\nu]/\alpha)}{\Gamma([1+\nu]/2)\Gamma(1-[1+\nu]/2)}\frac{(-1)^\nu}{\nu!}\left(\frac{|x|}{(K_\alpha t)^{1/\alpha}}\right)^\nu \tag{20}$$

derives. For $\alpha = 1$, the propagator reduces to the Cauchy LSD

$$P(x,t) = \frac{1}{\pi(K_1 t + x^2/[K_1 t])}. \tag{21}$$

We plot the time evolution of $P(x,t)$ for the Cauchy case $\alpha = 1$ in Fig. 1 in comparison to the limiting Gaussian case $\alpha = 2$.

Due to the point symmetry of the FFPE (15) for $F(x) = 0$, the propagator $P(x,t)$ is invariant under change of sign, and it is monomodal, i.e., it has its global maximum at $x = 0$, the point where the initial distribution $P(x,0) = \delta(x)$ was launched at $t = 0$. The latter property is lost in the case of strongly confined Lévy flights discussed below. Due to their Markovian character, Lévy flights also possess a Galilei invariance [22,8]. Thus, under the influence of a constant force

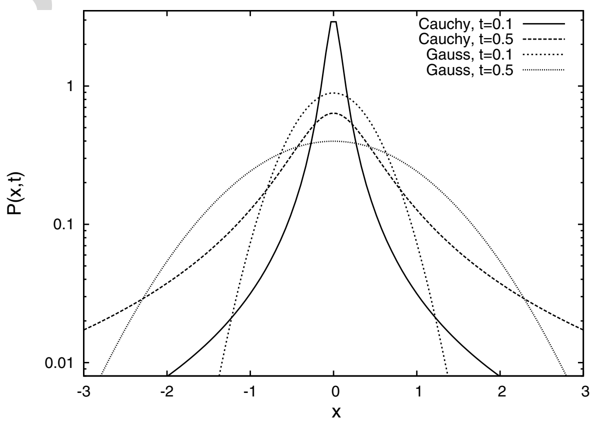

Fig. 1. Cauchy distribution ($\alpha = 1$) for two times in comparison to the Gaussian ($\alpha = 2$). We chose $K_1 = K_2 = 1$. Note that the Cauchy distribution is narrower at the origin, and after crossing the Gaussian falls off in the much slower power-law fashion.

field $F(x) = F_0$, the solution of the FFPE can be expressed in terms of the force-free solution by introducing the wave variable $x - F_0 t$, to obtain

$$P_{F_0}(x,t) = P_0\left(x - \frac{F_0 t}{m\gamma}, t\right). \tag{22}$$

This result follows from the FFPE (15), that in the Fourier domain becomes [19]

$$\frac{\partial}{\partial t}P(k,t) = \left(-\mathrm{i}k\frac{F_0}{m\gamma} - K_\alpha|k|^\alpha\right)P(k,t), \tag{23}$$

with solution

$$P(k,t) = \exp\left(-\left[\mathrm{i}k\frac{F_0}{m\gamma} + K_\alpha|k|^\alpha\right]t\right). \tag{24}$$

By the translation theorem of the Fourier transform, Eq. (22) yields. We show an example of the drift superimposed to the dispersional spreading of the propagator in Fig. 2.

4. Presence of external potentials

4.1. Harmonic potential

In an harmonic potential $V(x) = \frac{1}{2}\lambda x^2$, an exact form for the characteristic function can be found. Thus, from the corresponding FFPE in Fourier space,

$$\frac{\partial}{\partial t}P(k,t) = -\frac{\lambda}{m\gamma}k\frac{\partial}{\partial k}P(k,t) - K_\alpha|k|^\alpha P(k,t), \tag{25}$$

by the method of characteristics one obtains

$$P(k,t) = \exp\left(-\frac{m\gamma K_\alpha|k|^\alpha}{\lambda\alpha}\left[1 - \mathrm{e}^{-\alpha\lambda t/(m\gamma)}\right]\right) \tag{26}$$

for an initially central δ-peak, $P(x,0) = \delta(x)$ [19]. This is but the characteristic function of an LSD with time-varying width. For short times, $1 - \exp(-\alpha\lambda t/[m\gamma]) \sim \alpha\lambda t/[m\gamma]$ grows linearly in time, such that $P(k,t) \sim \exp(-K_\alpha|k|^\alpha t)$ as for a free Lévy flight. At long times, the stationary solution defined through

$$P_{\mathrm{st}}(k) = \exp\left(-\frac{m\gamma K_\alpha|k|^\alpha}{\lambda\alpha}\right), \tag{27}$$

is reached. Interestingly, it has the same stable index α as the driving Lévy noise. By separation of variables, a summation formula for $P(x,t)$ can be obtained similarly to the solution of the Ornstein–Uhlenbeck process in the presence of white Gaussian noise, however, with the Hermite polynomials replaced by H-functions [19].

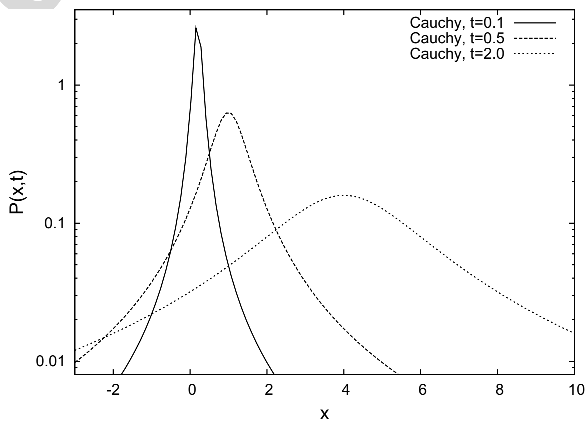

Fig. 2. Cauchy distribution with $K_1 = 1$ advected along a field $F_0/(m\gamma) = 2$, for different times.

We note that in the Gaussian limit $\alpha = 2$, the stationary solution by necessity has to match the Boltzmann distribution corresponding to $\exp(-k_B T k^2/[2\lambda])$. This requires that the Einstein–Stokes relation $K_2 = k_B T/[m\gamma]$ is fulfilled [23]. One might therefore speculate whether for a system driven by external Lévy noise a generalized Einstein–Stokes relation should hold, as was established for the subdiffusive case [10,8]. As will be shown now, in steeper than harmonic external potentials, the stationary form of $P(x,t)$ even leaves the basin of attraction of LSDs.

4.2. Steeper than harmonic potentials

To investigate the behaviour of Lévy flights in potentials, that are steeper than the harmonic case considered above, we introduce the non-linear oscillator potential

$$V(x) = \frac{a}{2}x^2 + \frac{b}{4}x^4, \tag{28}$$

that can be viewed as a next order approximation to a general confining, symmetric potential. It turns out that the resulting process differs from above findings if a suitable choice of the ratio a/b is made. For simplicity, we introduce dimensionless variables through

$$x \to x/x_0; \quad t \to t/t_0; \quad a \to at_0/(m\gamma), \tag{29}$$

where

$$x_0 = \left(\frac{m\gamma K_\alpha}{b}\right)^{1/(2+\alpha)}; \quad t_0 = \frac{x_0^\alpha}{K_\alpha}, \tag{30}$$

arriving at the FFPE

$$\frac{\partial}{\partial t}P(k,t) + |k|^\alpha = \left(k\frac{\partial^3}{\partial k^3} - ak\frac{\partial}{\partial k}\right)P(k,t). \tag{31}$$

Consider first the simplest case of a quadratic oscillator with $a = 0$ in the presence of Cauchy noise ($\alpha = 1$). In this limit, the stationary solution can be obtained exactly, yielding the expression

$$P_{st}(x) = \frac{1}{\pi}\frac{1}{1 - x^2 + x^4} \tag{32}$$

plotted in Fig. 3. Two distinct new features appear in comparison to the free Lévy flight, and the Lévy flight in an harmonic potential: (1) Instead of the maximum at $x = 0$, one observes two maxima positioned at

$$x_m = \pm\sqrt{1/2}; \tag{33}$$

at $x = 0$, we find a local minimum. (2) There occurs a power-law asymptote

$$P_{st}(x) \sim \frac{1}{\pi x^4} \tag{34}$$

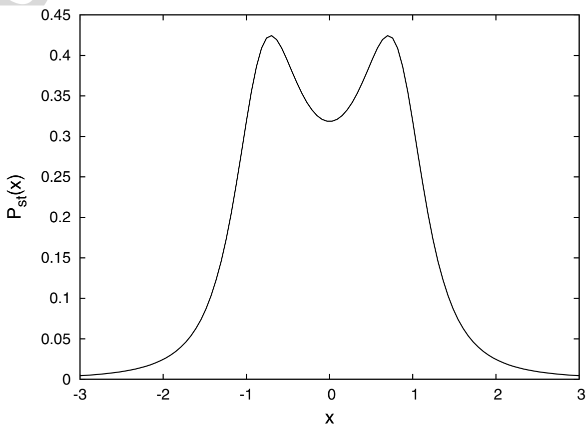

Fig. 3. Bimodal stationary probability density $P_{st}(x)$ from Eq. (32). The maxima are at $\pm\sqrt{1/2}$.

for $|x| \gg 1$; consequently, this stationary solution no longer represents an LSD, and the associated mean squared displacement is finite, $\langle x^2 \rangle < \infty$.

A more detailed analysis of Eq. (31) reveals [24,16], that (i) the bimodality of $P(x,t)$ occurs only if the amplitude of the harmonic term, a, is below a critical value a_c; (ii) for general α, the asymptotic behaviour is $P_{st}(x) \sim \pi^{-1} \sin(\pi\alpha/2)\Gamma(\alpha)|x|^{-\alpha-3}$; (iii) and there exists a finite bifurcation time t_c at which the initially monomodal form of $P(x,t)$ acquires a zero curvature at $x = 0$, before settling in the terminal bimodal form.

Interestingly, in the more general power-law behaviour

$$V(x) = \frac{|x|^c}{c}, \tag{35}$$

the turnover from monomodal to bimodal form of $P(x,t)$ occurs exactly when $c > 2$. The harmonic potential is therefore a limiting case when the solution of the FFPE still belongs to the class of LSDs and follows the generalized central limit theorem. This is broken in a superharmonic (steeper than harmonic) potential. The corresponding bifurcation time t_c is finite for all $c > 2$ [16]. An additional effect appears when $c > 4$: there exists a transient trimodal state when the relaxing $\delta(x)$-peak overlaps with the forming humps at $x = \pm x_m$. At the same time, the variance is finite, if only $c > 4 - \alpha$, following from the asymptotic stationary solution

$$P_{st}(x) \sim \frac{\sin(\pi\alpha/2)\Gamma(\alpha)}{\pi|x|^{\alpha+c-1}}. \tag{36}$$

Details of the asymptotic behaviour and the bifurcations can be found in Refs. [24,16]. From a reverse engineering point of view, Lévy flights in confining potentials are studied in [25].

5. Non-trivial boundary conditions

One might naively expect that a jump process of Lévy type, whose variance diverges (unless confined in a steep potential) may lead to ambiguities when boundary conditions are introduced, such as an absorbing boundary at finite x. Indeed, it is conceivable that for a jump process with extremely long jumps, it becomes ambiguous how to properly define the boundary condition: should the test particle be absorbed when it arrives exactly *at* the boundary, or when it crosses it *anyplace* during a non-local jump?

This question is trivial in the case of a narrow jump length distribution: all steps are small, and the particle cannot jump across a point (in the continuum limit considered herein). For such processes, one enforces a Cauchy boundary condition $P(0,t) = 0$ at the point $x = 0$ of the absorbing boundary, removing the particle once it hits the barrier after starting at x_0, where the dynamics is governed by Eq. (15) with $F(x) = 0$. Its solution can easily be obtained by standard methods, for instance, the method of images. This is completely equivalent to considering the *first arrival* to the point $x = 0$, expressed in terms of the diffusion equation with sink term:

$$\frac{\partial}{\partial t}\mathscr{P}(x,t) = K\frac{\partial^2}{\partial x^2}\mathscr{P}(x,t) - p_{fa}(t)\delta(x), \tag{37}$$

defined such that $P(0,t) = 0$. Note that the quantity $\mathscr{P}$ is no longer a probability density, as probability decays to zero; for this reason, we use the notation $\mathscr{P}$. From Eq. (37) by integration we obtain the survival probability

$$\mathscr{S}(t) = \int \mathscr{P}(x,t)\mathrm{d}x \tag{38}$$

with $\mathscr{S}(0) = 1$ and $\lim_{t\to\infty}\mathscr{S}(t) = 0$. Then, the first arrival density becomes

$$p_{fa}(t) = -\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{S}(t). \tag{39}$$

Eq. (37) can be solved by standard methods (determining the homogeneous and inhomogeneous solutions). It is then possible to express $\mathscr{P}(x,t)$ in terms of the propagator $P(x,t)$, the solution of Eq. (15) with $F(x) = 0$ with the same initial condition, $P(x,0) = \delta(x - x_0)$ and natural boundary conditions. One obtains

$$P(0,t) = \int_0^t p_{fa}(\tau)P(x_0, t-\tau)\mathrm{d}\tau, \tag{40}$$

such that the first arrival density corresponds to the waiting time distribution to jump from x_0 to 0 (or, vice versa, since the problem is symmetric). In Laplace space, this relation takes on the simple algebraic form $p_{fa}(u) = P(0,u)/P(x_0,u)$.

Both methods the explicit boundary value problem and the first arrival problem for Gaussian processes produce the well-known first passage (or arrival) density of Lévy–Smirnov type (1),

$$p(t) = p_{\rm fa}(t) = \frac{x_0}{\sqrt{4\pi K t^3}} \exp\left(-\frac{x_0^2}{4Kt}\right) \sim \frac{x_0}{\sqrt{4\pi K t^3}}, \tag{41}$$

with the asymptotic power-law decay $p(t) \sim t^{-3/2}$, such that no mean first passage time exists [4,26].

Long-tailed jump length distributions of Lévy stable form, however, endow the test particle with the possibility to jump across a certain point repeatedly. The first arrival necessarily becomes less efficient. Indeed, as shown in Ref. [27], the Gaussian result (41) is generalized to

$$p_{\rm fa}(t) \sim C(\alpha) \frac{x_0^{\alpha-1}}{K_\alpha^{1-1/\alpha} t^{2-1/\alpha}}, \text{as} \quad t \to \infty \tag{42}$$

with $C(\alpha) = \alpha\Gamma(2-\alpha)\Gamma(2-1/\alpha)\sin(\pi[2-\alpha]/2)\sin^2(\pi/\alpha)/(\pi^2[\alpha-1])$, and $1 < \alpha \geqslant 2$ [27]. The long-time decay $\sim t^{-2+1/\alpha}$ is slower than in (41). In Fig. 4, we show simulations results of the first arrival problem, corroborating the analytic result for various α. The insensitivity of the power-law slope to the initial condition is demonstrated in Fig. 5. However, we cannot realize a δ-point removal in the simulations but need to define a small but finite interval for removing the particle; Fig. 5 also shows that if this interval w is chosen too wide, for increasing width w the result slowly becomes steeper, eventually it should approach the $-3/2$ slope. Note that in both figures, we plot $tp_{\rm fa}(t)$ for clarity.

One might naively assume that the first passage problem (the particle is removed once it crosses the boundary) for Lévy flights should be more efficient, that is, the first passage density $p(t)$ should decay quicker, than for a narrow jump length distribution. However, as we have a symmetric jump length distribution $\lambda(\xi)$, the long outliers characteristic for these Lévy flights can occur both toward and away from the absorbing barrier. From this point of view it is not totally surprising to see the simulations result in Figs. 6 and 7, that clearly indicate a universal asymptotic decay $\sim t^{-3/2}$, exactly as for the Gaussian case.

In fact, for all Markovian processes with a *symmetric* jump length distribution, the Sparre Andersen theorem [28,2,29] proves without knowing any details about $\lambda(\xi)$ the asymptotic behaviour of the first passage time density universally follows $p(t) \sim t^{-3/2}$. The details of the specific form of $\lambda(\xi)$ only enter the prefactor, and the pre-asymptotic behaviour. A special case of the Sparre Andersen theorem was proved in Ref. [30] when the particle is released at $x_0 = 0$ at time $t = 0$, and after the first jump an absorbing boundary is installed at $x = 0$. This latter case was simulated extensively in Ref. [31]. From a fractional diffusion equation point of view, it was shown in Ref. [27] that the fractional operator $\partial^\alpha/\partial|x|^\alpha$ needs to be modified, to account for the fact that $\mathscr{P}(x,t) \equiv 0$ beyond the absorbing boundary, such that long-range correlations are present exclusively for all x in the semi-axis containing x_0. The fractional diffusion equation in the presence of the absorbing boundary therefore has to be modified to [27]

$$\frac{\partial}{\partial t}\mathscr{P}(x,t) = \frac{K_\alpha}{\kappa}\frac{\partial^2}{\partial x^2}\int_0^\infty \frac{\mathscr{P}(x',t)}{|x-x'|^{\alpha-1}}\mathrm{d}x', \tag{43}$$

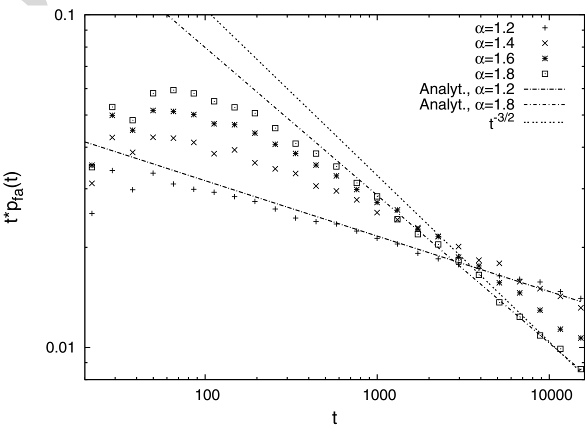

Fig. 4. First arrival density for various values of α and trap width $w = 0.25$. For $\alpha = 1.2$ and $\alpha = 1.8$ we compare to analytically predicted power-law (42). In all cases the decay is slower than for the Brownian result, $\sim t^{-3/2}$, that is also shown.

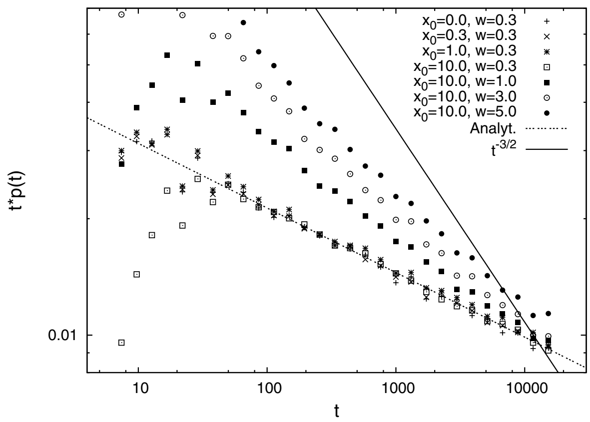

Fig. 5. First arrival density for $\alpha = 1.2$ for various initial positions x_0 and trap width $w = 0.3$, that turned out to be optimal in the simulations. All data follow the same scaling law predicted by Eq. (42). If w is chosen too wide, the slope increases toward the $t^{-3/2}$ behaviour. Note that we plot $tp_{\mathrm{fa}}(t)$, and that $p_{\mathrm{fa}}(t)$ is not normalized.

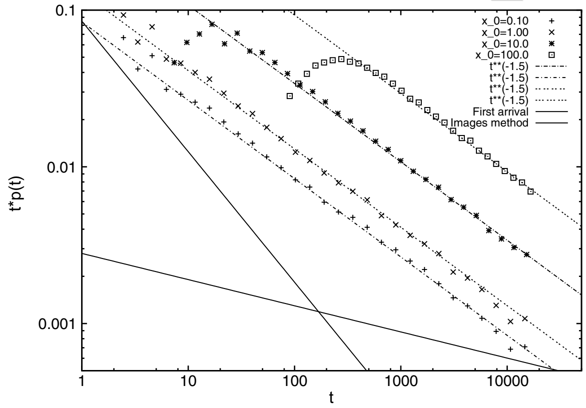

Fig. 6. First passage density for $\alpha = 1.2$ for various initial positions x_0 away from the absorbing boundary. Each time the particle crosses the boundary, it is removed. In all cases, the universal $\sim t^{-3/2}$ scaling is observed. The two additional lines represent the result of the corresponding first arrival problem and the images method.

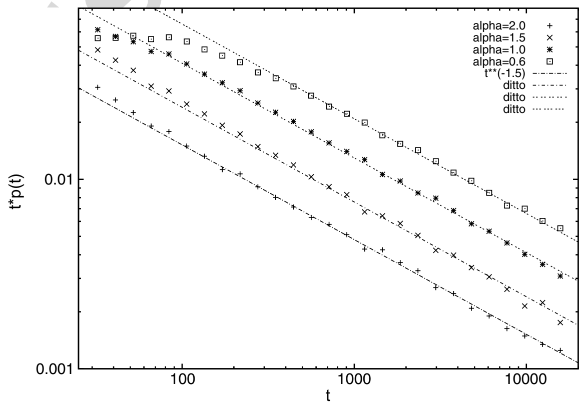

Fig. 7. First passage density for various stable indices α and initial position $x_0 = 10.0$ away from the absorbing boundary. Again, the universal $\sim t^{-3/2}$ scaling is distinct.

where $\kappa = 2\Gamma(2-\alpha)|\cos(\pi\alpha/2)|$, such that the first term on the right hand side no longer represents a Fourier convolution. An approximate solution with Cauchy boundary condition reveals $p(u) \sim 1 - cu^{1/2}$, where c is a constant, indeed leading to the Sparre Andersen behaviour $p(t) \sim t^{-3/2}$.

This also demonstrates that the method of images no longer applies when Lévy flights are considered, for the images solution

$$\mathscr{P}_{\text{im}}(x,t) = P(x - x_0, t) - P(x + x_0, t) \tag{44}$$

would be governed by the *full* fractional diffusion equation, and not Eq. (43), and the result for the first passage density, $p(t) \sim t^{-1-1/\alpha}$ would decay faster than the Sparre Andersen universal behaviour. A detailed discussion of the applicability of the method of images is given in terms of a subordination argument in Ref. [20]. We emphasize that this subtle failure of the method of images has been overlooked in literature previously [32,33], and care should therefore be taken when working with results based on such derivations. We also note that the method of images works in cases of subdiffusion, as the step length is narrow [34].

6. Kramers problem for Lévy flights

Many physical and chemical problems are related to the thermal fluctuations driven crossing of an energetic barrier, such as dissociation of molecules, nucleation processes, or the escape from an external, confining potential of finite height [35]. A particular example of barrier crossing in a double well potential driven by Lévy noise was proposed for a long-time series of paleoclimatic data [36]. Further cases where the crossing of a potential barrier driven by Lévy noise is of interest is in the theory of plasma devices [37], among others [9].

To investigate the detailed behaviour of barrier crossing under the influence of external Lévy noise, we choose the rather generic double well shape

$$V(x) = -\frac{a}{2}x^2 + \frac{b}{4}x^4. \tag{45}$$

Integrating the Langevin Eq. (18) with white Lévy noise, we find an exponential decay of the survival density in the initial well:

$$p(t) = \frac{1}{T_c}\exp\left(-\frac{t}{T_c}\right), \tag{46}$$

as demonstrated in Fig. 8. Lévy flight processes being Markovian, this is not surprising, since the mode relaxation is exponential [8,9]. More interesting is the question how the mean escape time T_c behaves as function of the characteristic noise parameters D and α. While in the regular Kramers problem with Gaussian driving noise the Arrhenius-type activation $T_c = C\exp(h/D)$ is followed, where h is the barrier height, and the prefactor C includes details of the potential, in the case of Lévy noise, a power-law form

$$T_c(\alpha, D) = \frac{C(\alpha)}{K_\alpha^{\mu(\alpha)}} \tag{47}$$

was assumed [38]. Detailed investigations [39] show that the scaling exponent $\mu(\alpha) = 1$ for all α strictly smaller than 2. As already proposed in Ref. [40] and derived in [41] in a somewhat different model, this means that, apart from a pre-

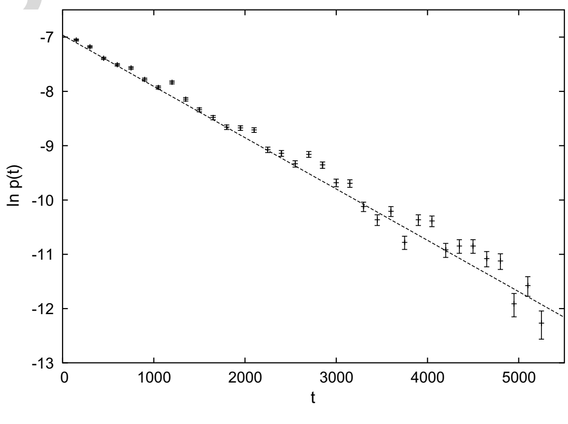

Fig. 8. Probability density function $p(t)$ of barrier crossing times for $\alpha = 1.0$ and $D = 10^{-2.5} \approx 0.00316$. The dashed line is a fit to Eq. (46) with MCT $T_c = 1057.8 \pm 17.7$.

factor, the Lévy flight is insensitive to the external potential for the barrier crossing. This behaviour is distinctly visible in Fig. 9 in form of the parallel lines in the log–log scale. Note that in comparison to Ref. [38], also values of α in the range $(0, 1)$ are included. For large values of D, deviations from the scaling are observed: eventually it will only take a single jump to cross the barrier when $D \to \infty$. Detailed studies show indeed that eventually the unit time step is reached, i.e., $T_c \to 1$.

7. More on the "pathology"

Despite their mathematical foundation due to the generalized central limit theorem and their broad use in the sciences and beyond as description for statistical quantities, and despite the existence of systems (for instance, the diffusion on a polymer in chemical space mediated by jumps where the polymer loops back on itself [42–44]), the divergence of the fluctuations of Lévy processes is sometimes considered a pathology. This was already put forward by West and Seshadri [45], who pointed out that a Lévy flight in velocity space would be equivalent to a diverging kinetic energy. Here, we show that higher order dissipation effects lead to natural cutoffs in Lévy processes.

At higher velocities the friction experienced by a moving body starts to depend on the velocity itself [46]. Such non-linear friction is known from the classical Riccati equation $M\,\mathrm{d}v(t)/\mathrm{d}t = Mg - Kv(t)^2$ for the fall of a particle of mass M in a gravitational field with acceleration g [47], or autonomous oscillatory systems with a friction that is non-linear in the velocity [46,48]. The occurrence of a non-constant friction coefficient $\gamma(V)$ leading to a non-linear dissipative force $-\gamma(V)V$ was highlighted in Klimontovich's theory of non-linear Brownian motion [49]. It is therefore natural that higher order, non-linear friction terms also occur in the case of Lévy processes.

We consider the velocity-dependent dissipative non-linear form (necessarily an even function) [50]

$$\gamma(V) = \gamma_0 + \gamma_2 V^2 + \cdots + \gamma_{2N} V^{2N} \quad \therefore \gamma_{2N} > 0 \tag{48}$$

for the friction coefficient of the Lévy flight in velocity space as governed by the Langevin equation

$$\mathrm{d}V(t) + \gamma(V)V(t)\mathrm{d}t = \mathrm{d}L(t) \tag{49}$$

with the constant friction $\gamma_0 = \gamma(0)$. $L(t)$ is the α-stable Lévy noise defined in terms of a characteristic function $p^*(\omega, t) = \mathscr{F}\{p(L, t)\} \equiv \int_{-\infty}^{\infty} p(L, t)\exp(\mathrm{i}\omega L)\mathrm{d}L$ of the form $p^*(\omega, t) = \exp(-D|\omega|^{\alpha} t)$ [1,51], where D of dimension $\mathrm{cm}^{\alpha}/\mathrm{s}$ is the generalised diffusion constant. This is equivalent to the fractional Fokker–Planck equation [15,37,16,8,9]

$$\frac{\partial P(V,t)}{\partial t} = \frac{\partial}{\partial V}(V\gamma(V)P) + D\frac{\partial^{\alpha} P}{\partial |V|^{\alpha}}. \tag{50}$$

As we showed in Section 4.2 by the example of the Lévy flights in position space, the presence of the first higher order correction, $\gamma_2 V^2$ in the friction coefficient $\gamma(V)$ rectifies the Lévy motion such that the asymptotic power-law becomes steeper and the variance finite. When even higher order corrections are taken into consideration, also higher order moments become finite. We show an example in Fig. 10 for the second moment.

The effect on the velocity distribution of the process defined by Eqs. (49) and (50) for higher order corrections are demonstrated in Fig. 11 for the stationary limit, $P_{\mathrm{st}}(V) = \lim_{t\to\infty} P(V, t)$: while for smaller V the character of the original

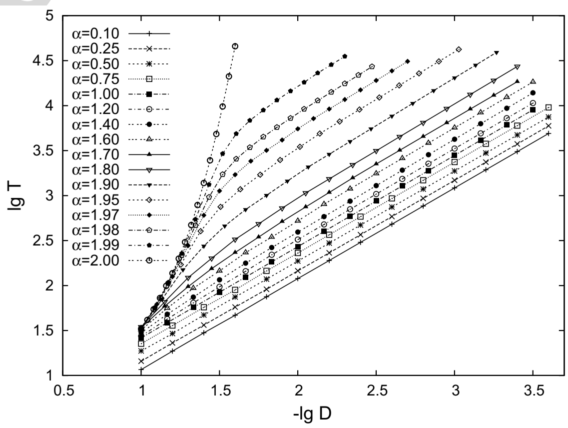

Fig. 9. Characteristic escape time as function of the diffusivity D for the double well potential.

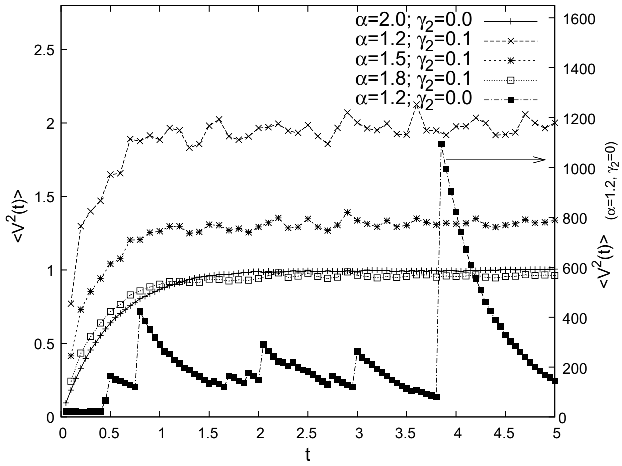

Fig. 10. Variance $\langle V^2(t)\rangle$ as function of time t, with the quadratic term set to zero, $\gamma_4 = 0$ and $\gamma_0 = 1.0$ for all cases. The variance is finite for the cases $\alpha = 2.0, \gamma_2 = 0.0$; $\alpha = 1.2, \gamma_2 = 0.1$; $\alpha = 1.5, \gamma_2 = 0.1$; and $\alpha = 1.8, \gamma_2 = 0.1$. These correspond to the left ordinate. For the case $\alpha = 1.2, \gamma_2 = 0.0$, the variance diverges, strong fluctuations are visible; note the large values of this curve corresponding to the right ordinate.

Lévy stable behaviour is preserved (the original power-law behaviour, that is, persists to intermediately large V), for even larger V the corrections due to the dissipative non-linearity are visible in the transition(s) to steeper slope(s).

These dissipative non-linearities remove the divergence of the kinetic energy from the measurable subsystem of the random walker. In the ideal mathematical language, the surrounding bath provides an infinite amount of energy through the Lévy noise, and the coupling via the non-linear friction dissipates an infinite amount of energy into the bath, and thereby introduces a natural cutoff in the kinetic energy distribution of the random walker subsystem. Physically, such divergencies are not expected, but correspond to the limiting procedure of large numbers in probability theory. We showed that both statements can be reconciled, and that Lévy processes are indeed physical.

Also Gaussian continuum diffusion exhibits non-physical features, possibly the most prominent being the infinite propagation speed inherent of the parabolic nature of the diffusion equation: even at very short times after system preparation in, say, a state $P(x,0) = \delta(x)$, there has already arrived a finite portion of probability at large x. This problem can be corrected by changing from the diffusion to the Cattaneo (telegrapher's) equation. Still, for most purposes, the uncorrected diffusion equation is used. Similarly, one often uses natural boundary conditions even though the system under consideration is finite, since one might not be interested in the behaviour at times when a significant portion of probability has reached the boundaries. In a similar sense, we showed that "somewhere out in the wings" Lévy flights are naturally cut off by dissipative non-linear effects. However, instead of introducing artificial cutoffs, knowing that for all purposes Lévy flights are a good quantitative description and therefore meaningful, we use "pure" Lévy stable laws in physical models.

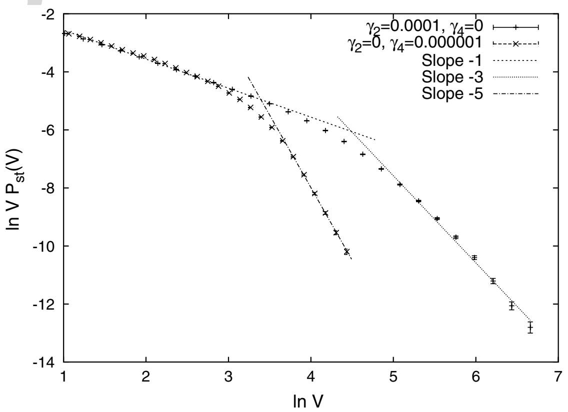

Fig. 11. Stationary PDF $P_{st}(V)$ for $\gamma_0 = 1.0$ and (i) $\gamma_2 = 0.0001$ and $\gamma_4 = 0$; and (ii) $\gamma_2 = 0$ and $\gamma_4 = 0.000001$; with $\alpha = 1.0$. The lines indicate the slopes -1, -3, and -5.

8. Discussion

Despite their popularity, comparatively long history, and their Markovian nature, Lévy flights are not completely understood. The proper formulation in the presence of non-trivial boundary conditions, their behaviour in external potentials both infinitely high and finite, as well as their thermodynamical meaning are under ongoing investigation. We here reported some important recent results.

While the continuous time random walk model for Lévy flights in the absence of non-trivial boundary conditions or external potentials is a convenient description, in all other cases the fractional Fokker–Planck equation or, equivalently, the Langevin equation with white Lévy stable noise are the description of choice. These equations in most cases cannot be solved exactly, however, it is usually straightforward to obtain the asymptotic behaviour, estimates for inflection points, etc., or to solve them numerically.

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