Leapover Lengths and First Passage Time Statistics for Lévy Flights

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Exact results for the first passage time and leapover statistics of symmetric and one-sided Lévy flights (LFs) are derived. LFs with a stable index α are shown to have leapover lengths that are asymptotically power law distributed with an index α for one-sided LFs and, surprisingly, with an index $\alpha/2$ for symmetric LFs. The first passage time distribution scales like a power law with an index $1/2$ as required by the Sparre-Andersen theorem for symmetric LFs, whereas one-sided LFs have a narrow distribution of first passage times. The exact analytic results are confirmed by extensive simulations.

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The statistics of first passage times is a classical concept to quantify processes, in which it is of interest when the dynamic variable crosses a given threshold value for the first time; e.g., when a tracer in some aquifer reaches a certain probe position, two molecules meet to form a chemical bond, animals search for sparse food locations, or a share at the stock market crosses a preset market value [1,2]. Here, we revisit the first passage time problem for processes with nontrivial jump length distributions, namely, Lévy flights (LFs), and derive exact asymptotic expressions for the first passage time density $p_f(\tau)$ of symmetric and one-sided LFs. For the former, we obtain the Sparre-Andersen universality $p_f(\tau) \simeq \tau^{-3/2}$, while a narrow behavior is found for one-sided LFs. Apart from calculating the first passage times, we investigate the behavior of the first passage leapovers, that is, the distance ℓ, the random walker overshoots the threshold value d in a single jump (see Fig. 1). Surprisingly, for symmetric LFs with a jump length distribution $\lambda(x) \simeq |x|^{-1-\alpha}$ (with index $0 < \alpha < 2$), the distribution of leapover lengths across $x = d$ is distributed like $p_l(\ell) \simeq \ell^{-1-\alpha/2}$; i.e., it is much broader than the original jump length distribution. In contrast, for one-sided LFs, the scaling of $p_l(\ell)$ bears the same index α.

For processes subject to a narrow jump length distribution with a finite second moment $\int_{-\infty}^{\infty} x^2\lambda(x)dx$, the crossing of a given threshold value d is identical to the first arrival at $x = d$ [2]. This is no longer true for LFs: Intuitively, a particle, whose jump lengths are distributed according to the symmetric long-tailed distribution $\lambda(x) \simeq |x|^{-1-\alpha}$ $(0 < \alpha < 2)$ is likely to crisscross the point $x = d$ multiple times before eventually hitting it, causing the first arrival at d to be slower than its first passage across d [3]. A measure for the ability to crisscross d is the distribution of leapover lengths, $p_l(\ell)$. Information on the leapover behavior of LFs is thus important to the understanding of how far proteins searching for their specific binding site along DNA overshoot their target [4], climatic forcing visible in ice core records exceeds a given value [5], or defining better stock market strategies determining when to buy or sell a certain stock instead of fixing a threshold price [6]. The quantification of leapovers is vital to estimate how far diseases would spread once a carrier of that disease crosses a certain border [7]. Leapover statistics of one-sided LFs provide an interesting alternative interpretation of the distribution of the first waiting time in ageing continuous time random walks [8], just to name a few examples.

The master equation for a Markovian diffusion process,

$$\frac{\partial P(x,t)}{\partial t} = \frac{1}{\tau}\int_{-\infty}^{\infty}[\lambda(x-x')P(x',t) - \lambda(x'-x)P(x,t)]dx', \tag{1}$$

accounts for the influx of probability to position x and the outflux away from x, where $\lambda(x)$ is a general, normalized jump length distribution. The time scale for single jumps is τ. The solution to Eq. (1) in Fourier space is $P(k,t) = e^{-[1-\lambda(k)]t/\tau}$, denoting the Fourier transform $f(k) = \int_{-\infty}^{\infty} e^{ikx}f(x)dx$ by explicit dependence on the wave number k. For instance, for the symmetric jump length distribution $\lambda(x) \simeq \sigma^\alpha|x|^{-1-\alpha}$, one finds

$$P(k,t) = e^{-K^{(\alpha)}|k|^\alpha t}, \tag{2}$$

with $K^{(\alpha)} = \sigma^\alpha/\tau$, the characteristic function of a symmetric Lévy stable law as obtained from continuous time

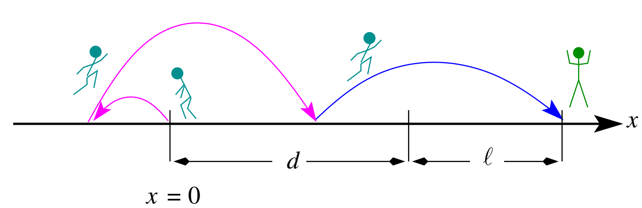

FIG. 1 (color online). Schematic of the leapover problem: the random walker starts at $x = 0$ and after a number of jumps crosses the point $x = d$, overshooting it by a distance ℓ.

random walk theory in the diffusion limit or from the equivalent space fractional diffusion equation [9].

In the following, we study processes with the long-tailed composite jump length distribution

$$\lambda(x)/\tau = \Theta(|x| - \varepsilon)[c_1\Theta(-x) + c_2\Theta(x)]/|x|^{1+\alpha}, \quad (3)$$

where $\Theta(x)$ is the Heaviside function. For $c_1 = c_2$, $\lambda(x)$ defines a symmetric LF, and for $c_1 = 0$ and $c_2 > 0$, a completely asymmetric (one-sided) LF permitting exclusively forward jumps. The cutoff ε excludes the singularity at $x = 0$, and we take $\varepsilon \to 0$ [10].

In the theory of homogeneous random processes with independent jumps, there exists a theorem, which provides an exact expression for the joint probability density function (PDF) $p(\tau, \ell)$ of first passage time τ and leapover length ℓ ($\ell \geq 0$) across $x = d$ for a particle initially seeded at $x = 0$ [11,12]. We here evaluate this theorem that appears to have been widely overlooked and derive a number of new analytic results for $p_f(\tau)$ and $p_l(\ell)$ of symmetric and one-sided LFs. With the probability to jump longer than x,

$$\mathcal{M}(x) = \int_x^\infty \lambda(x')dx', \qquad x > 0, \quad (4)$$

the theorem states that the double Laplace transform $p(u, \mu) = \int_0^\infty \int_0^\infty e^{-u\tau - \mu\ell} p(\tau, \ell) d\tau d\ell$ of the joint PDF is given in terms of the multiple integral [11,12]

$$p(u, \mu) = 1 - q_+(u, d) - \frac{\mu}{u} \times \int_0^d ds \int_{-\infty}^0 ds' \int_0^\infty ds'' \frac{\partial q_+(u, s)}{\partial s} \times \frac{\partial q_-(u, s')}{\partial s'} e^{-\mu s''} \mathcal{M}(d + s'' - s' - s). \quad (5)$$

The PDFs $p_f(\tau)$ and $p_l(\ell)$ for first passage time and leapover lengths follow from Laplace inversion of $p(u, 0)$ and $p(0, \mu)$, respectively. In Eq. (5), we use the auxiliary measures $q_\pm(u, x)$ defined through Fourier transforms

$$q_{x,\pm}(u, k) = \int_{-\infty}^\infty e^{ikx} \frac{\partial q_\pm(u, x)}{\partial x} dx = \exp\left\{\pm \int_0^\infty \frac{e^{-ut}}{t} \int_0^{\pm\infty} (e^{ikx} - 1) P(x, t) dx dt\right\}, \quad (6)$$

and the condition $q_\pm(u, 0) = 0$. They are related to the cumulative distributions of the maximum, $Q_+(t, d) = \Pr\{\max_{0\leq\tau\leq t} x(\tau) < d\}$, and minimum, $Q_-(t, d) = \Pr\{\min_{0\leq\tau\leq t} x(\tau) < d\}$, of the position $x(t)$ such that $q_\pm(u, d) = u \int_0^\infty e^{-ut} Q_\pm(t, d) dt$. The complicated integrals above reduce to elegant results for symmetric and one-sided LFs, as we show now.

For *symmetric LFs* ($c_1 = c_2 \equiv c$), the propagator is defined by the characteristic function (2) with generalized diffusion coefficient $K^{(\alpha)} = 2c\Gamma(1 - \alpha) \cos(\pi\alpha/2)/\alpha$. In the limit $u \to 0$ (long time limit), we obtain from Eq. (6)

$$q_{x,+}(u, k) \sim \frac{u^{1/2}}{\sqrt{K^{(\alpha)}}|k|^{\alpha/2}} \exp\left\{\frac{i\,\mathrm{sgn}(k)\pi\alpha}{4}\right\}. \quad (7)$$

Inverse Fourier transform and integration yields

$$q_+(u, d) \sim \frac{2u^{1/2}}{\alpha\sqrt{K^{(\alpha)}}\Gamma(\alpha/2)} d^{\alpha/2}, \qquad d > 0. \quad (8)$$

From $p_f(u) = 1 - q_+(u, d)$, we therefore find

$$p_f(\tau) \sim \frac{d^{\alpha/2}}{\alpha\sqrt{\pi K^{(\alpha)}}\Gamma(\alpha/2)} \tau^{-3/2} \quad (9)$$

for the asymptotic first passage time PDF valid for $\tau \gg d^\alpha/K^{(\alpha)}$ [13]. Figure 2 shows good agreement with the simulations [14]. We note that previously only the $\tau^{-3/2}$ scaling was known from simulations and application of Sparre-Andersen's theorem [3].

For symmetric LFs, for $0 < \alpha < 2$, we obtain that

$$\mathcal{M}(x) = \frac{K^{(\alpha)}}{2\Gamma(1 - \alpha)\cos(\pi\alpha/2)} x^{-\alpha}, \qquad x > 0. \quad (10)$$

Using that for symmetric LFs $q_-(\tau, x) = q_+(\tau, -x)$, it turns out after some transformations from Eq. (5) that

$$p_l(\mu) = \int_0^\infty e^{-\mu\ell} \frac{\sin(\pi\alpha/2)}{\pi} \frac{(d/\ell)^{\alpha/2}}{d + \ell} d\ell, \quad (11)$$

from which it follows immediately that

$$p_l(\ell) = \frac{\sin(\pi\alpha/2)}{\pi} \frac{d^{\alpha/2}}{\ell^{\alpha/2}(d + \ell)}, \quad (12)$$

see Fig. 3. Note that p_l is *independent* of $K^{(\alpha)}$. In the limit $\alpha \to 2$, $p_l(\ell)$ tends to zero if $\ell \neq 0$ and to infinity at $\ell = 0$ corresponding to the absence of leapovers in the Gaussian continuum limit. However, for $0 < \alpha < 2$, the leapover PDF follows an asymptotic power law with index $\alpha/2$

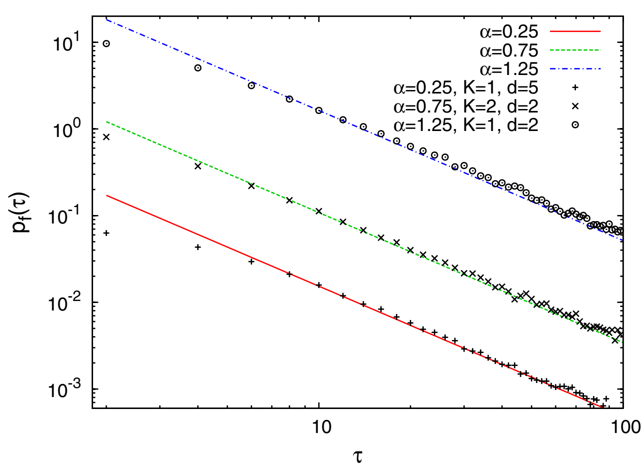

FIG. 2 (color online). First passage time density $p_f(\tau)$ for symmetric LFs. Lines represent Eq. (9). The curves for $\alpha = 0.75$ and 1.25 are multiplied by a factor of 10 and 100. Symbols: simulations.

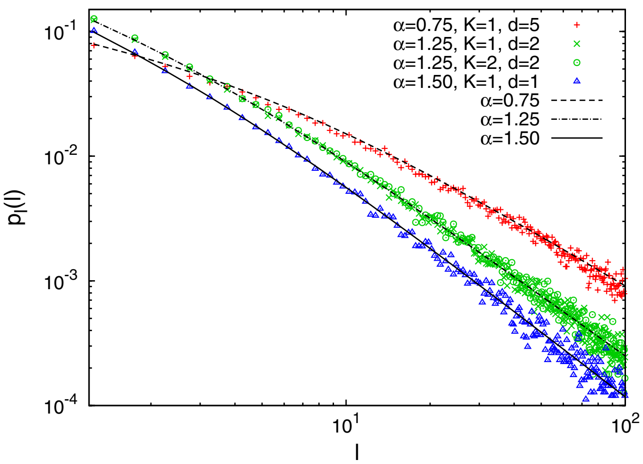

FIG. 3 (color online). Leapover density $p_l(\ell)$ for symmetric LFs. Lines according to the exact expression (12).

and is thus broader than the original jump length PDF $\lambda(x)$ with index α. This is remarkable: while λ for $1<\alpha<2$ has a finite characteristic length $\langle|x|\rangle$, the corresponding mean leapover length diverges.

Consider now *one-sided* LFs with $c_1=0$ in Eq. (3). In this case, the PDF has the characteristic function

$$P(k,t)=\exp\left\{-K^{(\alpha)}t|k|^\alpha\left[1-i\,\mathrm{sgn}(k)\tan\left(\frac{\pi\alpha}{2}\right)\right]\right\}, \quad (13)$$

where $K^{(\alpha)}=c_2\Gamma(1-\alpha)\cos(\pi\alpha/2)/\alpha$ and $\mathcal{M}(x)$ for $x>0$ is twice the expression in Eq. (10). Equation (6) leads to

$$q_{x,+}(u,k)=\frac{u}{u+\zeta}, \qquad \zeta=K^{(\alpha)}(-ik)^\alpha/\cos\left(\frac{\pi\alpha}{2}\right), \quad (14)$$

as $(-ik)^\alpha=[-i\,\mathrm{sgn}(k)|k|]^\alpha=|k|^\alpha\exp[-i\,\mathrm{sgn}(k)\pi\alpha/2]$. From this, we calculate

$$\int_{-\infty}^{\infty}e^{ikx}p_f(u)|_{d=x}dx=\frac{(-ik)^{\alpha-1}}{(-ik)^\alpha+u\cos(\pi\alpha/2)/K^{(\alpha)}}. \quad (15)$$

With the definition of the Mittag-Leffler function [9]

$$\int_0^\infty E_\alpha(-\theta x^\alpha)e^{-sx}dx=\frac{s^{\alpha-1}}{s^\alpha+\theta}, \quad (16)$$

and the substitution $ik\longrightarrow -s$, we obtain

$$p_f(u)=E_\alpha[-u\cos(\pi\alpha/2)d^\alpha/K^{(\alpha)}]. \quad (17)$$

From the relation between E_α and the M_α-function [15],

$$\int_0^\infty e^{-ut}M_\alpha(t)dt=E_\alpha(-u), \qquad 0<\alpha<1, \quad (18)$$

the following result for the first passage time PDF yields

$$p_f(\tau)=\frac{K^{(\alpha)}}{\cos(\alpha\pi/2)d^\alpha}M_\alpha\left(\frac{K^{(\alpha)}\tau}{\cos(\alpha\pi/2)d^\alpha}\right). \quad (19)$$

The M_α-function has the series representation and asymptotic behavior with exponential decay

$$M_\alpha(z)=\sum_{n=0}^\infty\frac{(-z)^n}{n!\Gamma(1-\alpha-\alpha n)} \quad (20)$$

$$\sim\frac{(\alpha z)^{(\alpha-1/2)/(1-\alpha)}}{\sqrt{2\pi(1-\alpha)}}\exp\left[-\frac{1-\alpha}{\alpha}(\alpha z)^{1/(1-\alpha)}\right]. \quad (21)$$

With $E_\alpha(-s)=1-s/\Gamma(1+\alpha)+\mathcal{O}(s^2)$ [9] and expansion $p_f(u)\sim 1-u\langle\tau\rangle$ of Eq. (17), the mean first passage time $\langle\tau\rangle=d^\alpha\cos(\pi\alpha/2)/[K^{(\alpha)}\Gamma(1+\alpha)]$ yields. $\langle\tau\rangle$ is *finite* and grows with the αth power of the distance d. For $\alpha=1/2$, we recover the exact form

$$p_f(\tau)=K^{(\alpha)}\sqrt{\frac{2}{\pi d}}\exp\left(-\frac{(K^{(\alpha)})^2\tau^2}{2d}\right). \quad (22)$$

$\langle\tau\rangle$ and Eq. (22) were previously obtained from a different method [16], while the full expression (19) for the PDF $p_f(\tau)$ has not been reported. The first passage PDF $p_f(\tau)$ is displayed in Fig. 4 in nice agreement with the simulations. Note that for $\alpha\le 1/2$, the tail of $\lambda(x)$ is so long that it is most likely to cross $x=d$ in the first jump, while for $\alpha>1/2$, $p_f(\tau)$ has a maximum at finite $\tau>0$.

To obtain the leapover statistics for the one-sided LF, we first note that since $P(x<0,t)=0$ (only forward steps are permitted), we have $q_{x,-}(u,k)=1$, and thus $\partial q_-(u,x)/\partial x=\delta(x)$. Combining Eqs. (5) and (6),

$$p_l(\mu)=1-\lim_{u\to 0}\frac{\mu}{u}\int_0^d\int_0^\infty e^{-\mu s'}\mathcal{M}(d+s'-s)\times\frac{\partial q_+(u,s)}{\partial s}ds'ds. \quad (23)$$

Expanding the Mittag-Leffler function, Eq. (17) produces

$$\frac{\partial q_+(u,x)}{\partial x}\sim\frac{u\cos(\pi\alpha/2)}{K^{(\alpha)}\Gamma(\alpha)}x^{\alpha-1}. \quad (24)$$

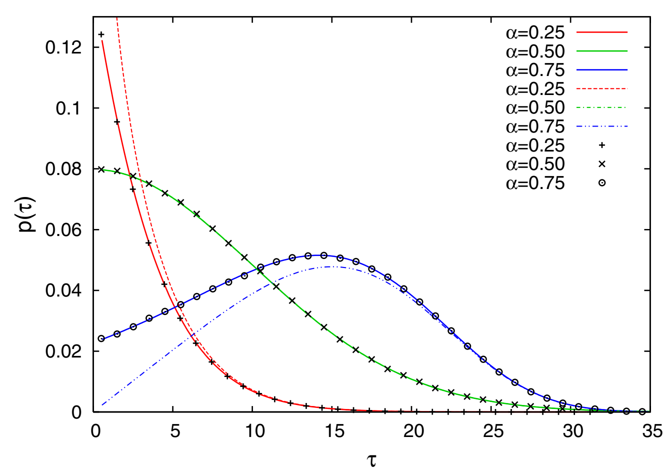

FIG. 4 (color online). First passage density for one-sided LF ($K^{(\alpha)}=1$). The full lines represent numerical evaluations using the exact analytic expression (20), while for the dashed lines, the asymptotic behavior (21) is used. Symbols: simulations.

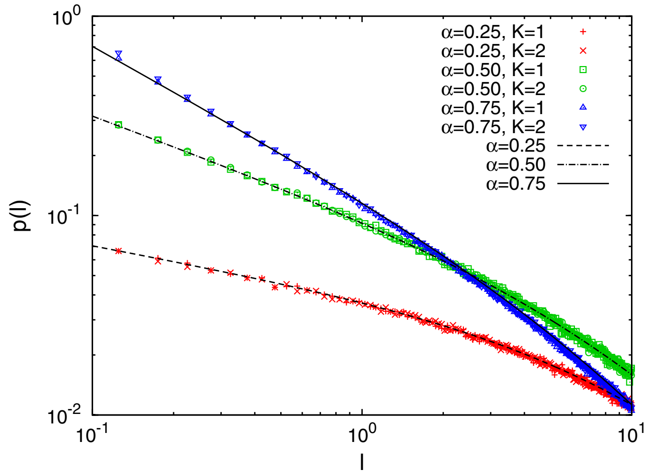

FIG. 5 (color online). Leapover distribution for one-sided LF with $d = 10$. Lines: exact asymptotic power law from Eq. (26).

Equations (14) and (24) inserted into Eq. (23) then yield

$$p_l(\mu) = \frac{\sin(\pi\alpha)}{\pi} \int_0^\infty e^{-\mu\ell} \frac{d^\alpha}{\ell^\alpha(d+\ell)}, \quad (25)$$

leading to the leapover PDF

$$p_l(\ell) = \frac{\sin(\pi\alpha)}{\pi} \frac{d^\alpha}{\ell^\alpha(d+\ell)}, \quad (26)$$

see Fig. 5, which corresponds to the result obtained in Ref. [16] from a different method. Thus, for the one-sided LF, the scaling of the leapover is exactly the same as for the jump length distribution, namely, with exponent α.

The leapover distribution (26) also provides a new aspect to the first waiting time in a renewal process with broad waiting time distribution $\psi(t) \simeq t^{-1-\beta}$ $(0 < \beta < 1)$. Interpret the position x as time and the jump lengths drawn from the one-sided $\lambda(x)$ as waiting times t. Consider an experiment, starting at time t_0, on a system prepared at time 0 (corresponding to position $x = 0$). Then, the first recorded waiting time t_1 of the system will be distributed like $p_1(t_1) = \pi^{-1}\sin(\pi\alpha)t_0^\alpha/[t_1^\alpha(t_0 + t_1)]$, as obtained from a different reasoning in Ref. [8]. We note that the first passage time τ in this analogy corresponds to the number of waiting events.

While for symmetric LFs, it was previously established that the first passage time distribution follows the universal Sparre-Andersen asymptotics $p_f(\tau) \simeq \tau^{-3/2}$; here, we derived for the first time the prefactor of this law, in particular, its dependence on the generalized diffusion coefficient $K^{(\alpha)}$. For the same case, we derived the previously unknown leapover distribution $p_l(\ell)$, which is interesting for two reasons: (i) $p_l(\ell)$ is independent of $K^{(\alpha)}$, synonymous to the noise strength; (ii) its power law exponent is $\alpha/2$, and thus $p_l(\ell)$ is broader than the original jump length distribution. For one-sided LFs, we found the previously reported leapover distribution and derived the so far unknown first passage time distribution, whose first moment was derived from a different method before. While the leapovers follow the same asymptotic scaling $p_l(\ell) \simeq \ell^{-1-\alpha}$ as the jump lengths $\lambda(x)$, once more independent of $K^{(\alpha)}$, the first passage times are narrowly distributed. We also drew an analogy between the leapovers and the first waiting time in a subdiffusive renewal process. For both symmetric and one-sided LFs, extensive simulations confirmed the analytic results.

Knowledge of the prefactors of the leapover and first passage distributions, the dependence on distance d and leapover length ℓ in particular, will be useful for comparison with experimental data, e.g., to describe threshold or target overshoot properties in search problems, climate records, stock market prices, or disease spreading [4–7].

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