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LETTER TO THE EDITOR

Non-uniqueness of the first passage time density of Lévy random processes

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Abstract
We obtain the first passage time density for Lévy random processes (LRPs) from a subordination scheme, demonstrating that the first passage time density cannot be inferred uniquely from the probability density function $P(x, t)$ governing the random process. This is due to the fact that $P(x, t)$ does not contain all information on the trajectory of the underlying LRP.

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An important issue in the theory of stochastic processes is the problem of first passage [1–4]: its solution is a key to the understanding of chemical reactions, stability of states of dynamical systems under external perturbations, extinction of populations, and many other problems in natural sciences. In normal Fickian diffusion, knowledge of the probability density function (PDF) of the process (Green's function) sufficiently defines the first passage time density (FPTD); namely, the FPTD directly follows from the renewal property of the Markovian process [2, 4]. More practically, the same information arises from the solution of the corresponding boundary value problem of the diffusion equation, or from the method of images due to Kelvin [1, 4, 5]. However, for Lévy flights (LFs), i.e., Markovian random processes with long-tailed jump lengths $\lambda(x) \sim \sigma^{\mu}|x|^{1+\mu}$ $(0 < \mu < 2)$ [6], it has recently been demonstrated that the images method leads to a result that contradicts the Sparre–Andersen theorem, according to which the FPTD of a random walk process asymptotically follows the universal $f(t) \sim t^{-3/2}$ behaviour for any symmetric distribution of jump lengths [1, 4, 7, 8]. Here, we consider more general (than LFs) random processes that possess a long-tailed $\lambda(x)$, but are not simple jump processes as functions of a clock time t ('laboratory time'). This general class of processes will be called Lévy random processes (LRPs). As an example we consider a class of LRPs subordinated (in a sense defined below) to LFs or to Brownian random walks. We show that the PDF of an LRP does not uniquely describe the FPTD, since the same PDF can correspond to LRPs with different sample path fractal dimensions. Hence, the exponent of the asymptotic form of the FPTD is governed by an inequality with a lower bound corresponding to the Sparre–Andersen result for a pure jump process, and an upper bound given by the result of the images method. We find that the latter corresponds to a process subordinated to a Fickian diffusion (Wiener) process. Consequently, it is necessary to have detailed knowledge of the properties of the trajectory of the underlying random process to uniquely determine the FPTD.

LRPs, for instance, LFs, are a paradigm for anomalous stochastic processes with a wide range of applications such as chaotic dynamics, processes in plasma, transport in micelles, or even quantum systems [9–15]. LFs can be considered as appearing from a Langevin equation with δ-correlated Lévy noise [16–18], the characteristic function of an LF being given by

$$P(k,t) = \exp(-K^{(\mu)}|k|^{\mu}t) \tag{1}$$

of the stretched Gaussian type. $P(k,t) \equiv \mathcal{F}\{P(x,t)\} = \int_{-\infty}^{\infty} P(x,t)\exp(\mathrm{i}kx)\,\mathrm{d}x$ is the Fourier transform of the PDF $P(x,t)$ [3, 9, 10, 19]. In what follows, the generalized diffusion constant $K^{(\mu)}$ will be set unity. In the limit $\mu = 2$, characteristic function (1) reduces to the Gaussian $P(k,t) = \exp(-K^{(2)}k^2t)$, the characteristic function of a standard random walk with a Gaussian limit distribution and the finite variance $\langle x^2(t)\rangle = 2K^{(2)}t$ [2, 20], as well as finite higher-order moments. For the general case $0 < \mu < 2$, only fractional moments of the form

$$\langle |x|^{\delta}\rangle = \frac{2^{1+\delta/\mu}}{\mu\pi^{1/2}}\frac{\Gamma(1/2+\delta/2)\Gamma(-\delta/\mu)}{\Gamma(-\delta/2)}(K^{(\mu)}t)^{\delta/\mu}, \tag{2}$$

exist with $\delta < \mu$ [21]. From the Langevin equation with Lévy noise follows the fractional diffusion equation [11, 12, 17, 21]. With its non-local fractional Riesz–Weyl operator replacing the Fickian second-order space derivative, this fractional equation reflects the absence of the variance $\langle x^2(t)\rangle$ [11, 18, 21, 22].

For pure LFs, i.e., for pure jump processes with the PDF given by equation (1) and fractal dimension $d_f = \mu$ of the sample path, the method of images leads to results that are inconsistent with the universal Sparre–Anderson asymptotic $f(t) \sim t^{-3/2}$ for the FPTD [8]; namely, the method of images leads to the contradictory result that the FPTD explicitly depends on the Lévy index μ. The failure of the method of images for this LF-process is due to the fact that the probability density of first arrival at a site differs from the density of first passage, the former being explicitly dependent on the index μ of the LF [8].

In what follows, we present an alternative derivation of the FPTD for general LRPs based on a subordination to a regular random walk process along the lines of [23, 24], leading us to a new, *a priori* unexpected twist: although the PDF of the free processes (without boundaries) is the same (with characteristic function (1)), this PDF does not uniquely describe the whole process and is consistent with different forms of the FPTD, depending on the exact properties of the underlying sample paths.

Let us consider a stochastic process subordinated to a discrete random walk with some operational time that is defined by a one-sided Lévy law [23]. The notion of subordination implies that the corresponding random process can be understood as follows [1]: the motion of the random walker can be parametrized by the number of steps n, the PDF of the random walker's position after n steps being given by the PDF $P_{\mathrm{RW}}(x,n)$. The corresponding characteristic function $\varphi_n(k) = \int_{-\infty}^{\infty} P_{\mathrm{RW}}(x,n)\exp(\mathrm{i}kx)\,\mathrm{d}x$ is determined by the PDF of step lengths $\lambda(x)$ such that $\varphi_n(k) = [\lambda(k)]^n$ in terms of the characteristic function $\lambda(k)$ of the step length PDF; compare, for instance [3].

The number of steps n itself, which may be considered as the intrinsic (operational) time of the process, is a nondecaying random function of the physical time (clock time) t. Denoting by $p_n(t)$ the probability of performing exactly n steps up to time t, we obtain

$$P(x,t) = \sum_n P_{\mathrm{RW}}(x,n)p_n(t). \tag{3}$$

This relation describes a large class of random processes: if the typical value of n grows slower than linearly in t, equation (3) defines the PDF of subdiffusive continuous-time random walks; if this value grows faster than linearly in t, the overall process would be superdiffusive [23]. At long times we suppose that a continuous operational time T may be introduced instead of the discrete index n. Moreover, we assume that the continuous analogue of $P_{\mathrm{RW}}(x,n)$ exists. In this limit, we find from (3)

$$P(x,t) = \int_0^\infty P_x(x,T)p_T(T,t)\,\mathrm{d}T. \tag{4}$$

$P_x(x,T)$ is the PDF to be at x at the operational time T and $p_T(T,t)$ is the PDF to be at the operational time T at the clock time t. This continuous limit corresponds exactly to the mathematical notion of subordination, and we assume $P(x,t)$ is subordinated to $P_x(x,T)$.[3]

Instead of having the PDF $P_{\mathrm{RW}}(x,n)$ of a Brownian random walk, let us assume the more general case that $P_{\mathrm{RW}}(x,n)$ corresponds to an LF, so that $P_{\mathrm{RW}}(x,n)$ is given by a symmetric Lévy distribution of the form[4]

$$P_{\mathrm{RW}}(x,n) = (l_0 n^{1/\alpha})^{-1} L\left(\frac{x}{l_0 n^{1/\alpha}}, \alpha, 0\right) \tag{5}$$

($0 < \alpha \leqslant 2$), where l_0 is a scaling factor with the dimension of length. Note that the limit $\alpha = 2$ corresponds to a Gaussian profile for $P_{\mathrm{RW}}(x,n)$. Let us additionally assume that the number of steps per unit of the physical (clock) time t is distributed according to some distribution with a power-law tail: $p(n, \Delta t = 1) \propto n^{-1-\beta}$, with $0 < \beta \leqslant 1$. Then, according to the generalized central limit theorem, at long times $p_n(t)$ tends to a continuous limit distribution corresponding to the one-sided Lévy law

$$p_T(T,t) = \left(\frac{\tau_0}{t}\right)^{1/\beta} L\left(T\left(\frac{\tau_0}{t}\right)^{1/\beta}; \beta, -\beta\right). \tag{6}$$

Here, τ_0 is a scaling factor with the dimension of time. The one-sided character of (6) ensures that $p_T(T < 0, t) \equiv 0$ [1, 19].

To obtain the limit distribution $P(x,t)$ based on relations (5) and (6), we first Fourier-transform equation (4) with respect to x. With expression (1) for the characteristic function of a symmetric Lévy stable density, we find

$$P(k,t) = \int_0^\infty \exp(-|kl_0|^\alpha T)\left(\frac{\tau_0}{t}\right)^{1/\beta} L\left(T\left(\frac{\tau_0}{t}\right)^{1/\beta}; \beta, -\beta\right)\mathrm{d}T. \tag{7}$$

This exactly corresponds to the Laplace transform of a one-sided Lévy stable density with the Laplace variable $u = |k|^\alpha l_0^\alpha$. The Laplace transform of a one-sided Lévy stable density is known [3, 19]: $\tilde{p}_T(u,t) = \exp(-u^\beta t/\tau_0)$, and thus

$$P(k,t) = \exp\left(-|k|^{\alpha\beta}\left[l_0^{\alpha\beta}\tau_0^{-1}\right]t\right). \tag{8}$$

[3] Compare section X.7 of Feller [1]: 'if $\{X(T)\}$ is a Markov process with continuous transition probabilities and $\{T(t)\}$ a process with non-negative independent increments, then $\{X(T(t))\}$ is said to subordinate to $\{X(t)\}$ using the operational time T.'

[4] Here, we use the canonical notation $L(x,\alpha,\gamma)$ for Lévy stable densities [1]. Thus, $L(x,\alpha,0)$ denotes symmetric Lévy distributions, and $L(x,\alpha,-\alpha)$ the extreme asymmetric Lévy stable densities. Defined for $0 < \alpha < 1$, one-sided Lévy laws $L(x,\alpha,-\alpha)$ vanish identically for $x < 0$.

This expression, in turn, is the characteristic function of a symmetric Lévy stable density with index $\alpha\beta$. The scaling factor $K^{(\alpha\beta)} \equiv l_0^{\alpha\beta}/\tau_0$ can be interpreted as the associated fractional diffusion coefficient that depends on the indices of the corresponding subprocesses only through their product $\mu = \alpha\beta$.

The characteristic function (8) defines a symmetric Lévy stable density with index μ, and fulfils the fractional diffusion equation of the same order. This can also be shown on the grounds of the subordination scheme developed in [23, 25]. However, it is remarkable that the PDF $P(x,t)$ is not specific to a unique random process, unless we have to deal with the limiting case $\mu = 2$ that necessarily corresponds to $\alpha = 2$ and $\beta = 1$, defining the process unambiguously. In other words, although the PDFs of all processes with identical μ are the same, these processes may still differ in the fractal dimension of their sample paths given by the set of jumps of random length corresponding to LFs with index α; and similarly they differ in the nature of the connection between the operational time T and the physical clock time t according to the one-sided Lévy stable density with index β. This issue is vital for the behaviour of associated first passage problems, as we now demonstrate.

Our subordination procedure corresponds to a (random) change of the time variable of the process from the operational time T to the clock time t. This allows us to solve a number of problems connected to the underlying random process without explicitly referring to the associated fractional equation, whose explicit form in the presence of an absorbing boundary condition is expected to differ from its well-established form in infinite space [8]. We consider the first passage across a boundary located at $x = 0$.

Let $\mathcal{S}_n(x_0)$ be the survival probability on the positive semi-axis (i.e. the probability of not crossing the boundary within the first n steps) after starting at $x_0 > 0$ at $n = 0$. According to the Sparre–Andersen theorem [1, 4, 7], the asymptotic form of this probability does not depend on the jump length distribution if only it is symmetric. For a large number of steps, one invariably has $\Psi_n(x_0) \simeq c(x_0)n^{-1/2}$, where the prefactor $c(x_0)$ depends on the initial position x_0, as well as on α and l_0. On subordinating the number of steps n to the physical clock time t we see that the survival probability up to time t is

$$\mathcal{S}(t; x_0) = \sum_n \Psi_n(x_0) p_n(t), \tag{9}$$

where $p_n(t)$ is the probability of occurring exactly n steps within the clock time t. Changing from n to the continuous operational time variable T, we get $\mathcal{S}(t; x_0) = \int_0^\infty \Psi(T; x_0) p_T(T,t)\,\mathrm{d}T$. This and $\Psi_n(x_0) \simeq c(x_0)n^{-1/2}$ give rise to

$$\mathcal{S}(t; x_0) \simeq c(x_0)\frac{1}{\pi^{1/2}\beta}\Gamma\left(\frac{1}{2\beta}\right)\left(\frac{\tau_0}{t}\right)^{1/(2\beta)}, \tag{10}$$

as derived in the appendix. We now obtain the FPTD,

$$f(t) = -\frac{\mathrm{d}\mathcal{S}(t; x_0)}{\mathrm{d}t} \simeq \frac{c(x_0)}{\pi^{1/2}\beta}\Gamma\left(1 + \frac{1}{2\beta}\right)\frac{\tau_0^{1/(2\beta)}}{t^{1+1/(2\beta)}}. \tag{11}$$

The following limiting cases can be distinguished:

(i) If the subordination from the operational time T to the clock time t through $p_T(T,t)$ is narrow with $\beta = 1$, i.e., $p_T(T,t) = \delta(T - t/\tau_0)$, the universal $f(t) \sim t^{-3/2}$ behaviour according to the Sparre–Anderson theorem is recovered. In other words, in order to change this asymptotic behaviour, one has to consider $0 < \beta < 1$ explicitly.

(ii) If we consider the process subordinated to the Gaussian diffusion ($\alpha = 2$), then $\beta = \mu/2$ and

$$f(t) \simeq \frac{2c(x_0)}{\pi^{1/2}\mu}\Gamma(1 + \mu)\frac{\tau_0^{1/\mu}}{t^{1+1/\mu}}. \tag{12}$$

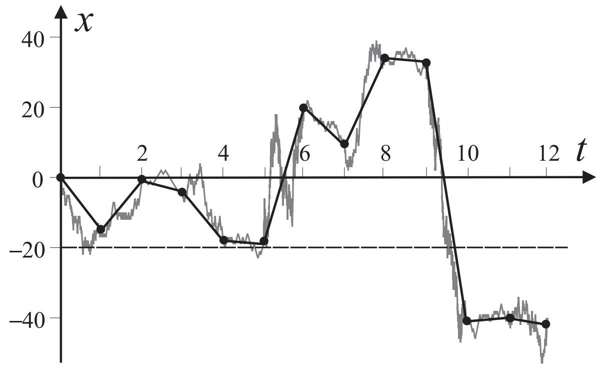

Figure 1. Trajectory of a Cauchy-LRP (dots and full lines) subordinated to a Brownian motion (grey), as obtained by sampling the Brownian trajectory at integers of t. The exact generation of such trajectories is discussed in [24].

This result has the same scaling as the FPTD derived through the method of images in [8]. A few words on the interpretation of this seemingly paradox finding are in order. Result (12) corresponds to a random walk process with a Gaussian jump length density $\lambda(x)$, so that the corresponding trajectory is that of a regular Brownian walk (with the fractal dimension $d_f = 2$ of the sample path). It is therefore perfectly legitimate to use the images method for such a process, even though it has the same PDF as an LF with the same order μ. In contrast, the latter, genuine LF discussed in [8] corresponds to a broad $\lambda(x) \sim l_0^\alpha \big/ |x|^{1+\alpha}$ with $\alpha < 2$, but $\beta = 1$. For this strongly non-local jump process with the trajectory fractal dimension $d_f = \mu$, the FPTD follows the result $f(t) \sim t^{-3/2}$; that is, the method of images fails.

(iii) In general, for a given μ one has to make sure that the inequalities $\mu/2 < \beta \leqslant 1$ are fulfilled, since simultaneously the two conditions $\beta \leqslant 1$ and $\beta = \mu/\alpha$ with $0 < \alpha \leqslant 2$ have to be met. The FPTD for such a general μ therefore shows the following asymptotic behaviour $f(t) \propto t^{-\delta}$, $3/2 \leqslant \delta \leqslant 1+1/\mu$, where $0 < \mu \leqslant 2$. In this scheme, the Sparre–Andersen decay with exponent 3/2 is the slowest one possible. This makes perfect sense since due to the Lévy stable form of $p_n(t) \sim n^{-1-\beta}$ a broad distribution of single-jump events occurs in a finite time interval $(0, t)$, increasing the likelihood of crossing the boundary within any given finite time interval dramatically. The Lévy nature of $p_n(t)$ thus leads to an oversampling of the space in comparison to the process $\beta = 1$.

The situation is illustrated in figure 1, showing a trajectory of a Cauchy-LRP. This LRP is subordinated to a Brownian motion (shown in grey) corresponding to $\alpha = 2$ and $\beta = 1/2$ by the law $X(T(t))$ (with $X(T)$ being a Wiener process). The trajectory of the Cauchy-LRP classified by $\alpha = 1, \beta = 1$ is obtained by sampling the Brownian trajectory at times $t = 0, 1, 2, \ldots,$ as shown by connecting the black dots. The boundary at $x = -20$ is crossed by the Brownian process at some $t < 1$, while the corresponding subordinated process crosses the boundary only in its 10th step. The difference between the processes also persists asymptotically, leading to the fact that the pure jump process corresponds to the slowest decay of the FPTD at a longer t.

The subordination scheme developed here allows one to express the FPTD of a random process solely on the basis of the properties of the Sparre–Andersen universality and the subordination from the operational time T to the physical clock time. Starting off from an

LF in (n, t) coordinates, we introduce a broad distribution $p_n(t) \sim n^{-1-\beta}$ of events per clock time interval Δt. We find that the resulting process follows a Lévy stable density of order $\mu = \alpha\beta$, the latter being a product of the Lévy index of the jump length distribution, α, and the subordination distribution $p_n(t)$, β. Knowledge of μ alone is therefore insufficient to deduce the exact form of the jump length PDF and the trajectory it gives rise to.

As a direct consequence, the resulting FPTD in the limit $p_n = \delta_{n,1}$ (the trivial subordination with $\beta = 1$) fulfils the Sparre–Anderson universality for any process with a symmetric jump length distribution. This case includes the case of genuine LFs as those discussed in [8], and is violated by the method of images for all $0 < \alpha < 2$. Conversely, for $\alpha = 2$, the subordination process has the trajectory of a normal Brownian random walk and is amenable to the images method to determine the FPTD. The result scales like $f(t) \sim t^{-1-1/\mu}$ as previously obtained. In the case of general α and β, the range spanned by the exponent in the FPTD $f(t) \sim t^{-1-\delta}$ is $3/2 \leqslant \delta \leqslant 1 + 1/\mu$.

We believe that above findings help interpret the inadequacy of the images method for genuine LFs as found in [8] and, moreover, show that caution is necessary when generalizing well-known results from ordinary to anomalous diffusive processes. Since the Lévy stable PDF with characteristic function (1), described dynamically by the fractional diffusion equation, governs all LRPs in the subordination sense defined above equally, to infer the complete properties (such as the FPTD) of the process requires, in addition, knowledge of the nature of the trajectory of the LRP. It is therefore of interest to obtain a full description for all types of LRPs with the full boundary value problem and trajectory information.

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Appendix. Calculation of the $(-1/2)$-order moment

To obtain the final form of equation (10), we have to evaluate the integral

$$\int_0^\infty T^{-1/2} \left(\frac{\tau_0}{t}\right)^{1/(2\beta)} L\left(T\left(\frac{\tau_0}{t}\right)^{1/\beta}; \beta; -\beta\right), \tag{13}$$

which is equal to the $(-1/2)$-order moment $M_{-1/2}(\beta) = \int_0^\infty \xi^{-1/2} L(\xi;\beta;-\beta)\,\mathrm{d}\xi$ of the one-sided Lévy stable density $L(\xi;\beta;-\beta)$. The latter is given by the inverse Laplace transform of a stretched exponential, which can be expressed in terms of the Fox H-function $\exp(-u^\beta) = \beta^{-1} H^{1,0}_{0,1}\left[u\,\middle|\,{}^{-;-}_{(0,1/\beta)}\right]$ [21, 26]. By standard methods [26, 27], one obtains

$$L(\xi;\beta;-\beta) = (\beta\xi)^{-1} H^{1,0}_{1,1}\left[\frac{1}{\xi}\,\middle|\,\begin{matrix}(0,1)\\(0,1/\beta)\end{matrix}\right]. \tag{14}$$

After a substitution, the resulting integral

$$M_{-1/2}(\beta) = \frac{1}{\beta}\int_0^\infty z^{-1/2} H^{1,0}_{1,1}\left[z\,\middle|\,\begin{matrix}(0,1)\\(0,1/\beta)\end{matrix}\right] \tag{15}$$

can be evaluated, by noting that it corresponds to the Mellin transform $\hat{g}(s) \equiv \int_0^\infty t^{s-1} g(t)$ of $H^{1,0}_{1,1}(z)$ at $s = 1/2$. The result can be identified with the definition of the H-function [21, 26]: $M_{-1/2}(\beta) = \beta^{-1}\Gamma(1/[2\beta])/\Gamma(1/2)$. This reproduces equation (10).

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