Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
Chemical Physics 284 (2002) 67–90
Chemical Physics
www.elsevier.com/locate/chemphys

Space- and time-fractional diffusion and wave equations, fractional Fokker–Planck equations, and physical motivation

Ralf Metzler [a,b,*], Theo F. Nonnenmacher [c]

[a] Department of Physics, Massachusetts Institute of Technology, 77 Massachusetts Avenue, Cambridge, MA 02139, USA
[b] NORDITA, Blegdamsvej 17, DK-2100 København Ø, Denmark
[c] Department of Mathematical Physics, University of Ulm, Albert-Einstein-Allee 11, 89069 Ulm/Donau, Germany

Received 29 October 2001

Abstract

We investigate the physical background and implications of a space- and time-fractional diffusion equation which corresponds to a random walker which combines competing long waiting times and Lévy flight properties. Explicit solutions are examined, and the corresponding fractional Fokker–Planck–Smoluchowski equation is presented. The framework of fractional kinetic equations which control the systems relaxation to either Boltzmann–Gibbs equilibrium, or a far from equilibrium Lévy form is explored, putting the fractional approach in some perspective from the standard non-equilibrium dynamics point of view, and its generalisation.

PACS: 05.40.Fb; 05.60.Cd; 02.50.Ey

Keywords: Fractional diffusion equation; Fractional Fokker–Planck equation; Anomalous diffusion; Lévy flights; Lévy walks

1. Introduction

A cornerstone in the development of modern physics was the introduction of the concept of probability into atomistic physics in Maxwell's theory of gas kinetics [1], and in Boltzmann's transport equation [2]. In these theories, particles are usually viewed, from a stochastic standpoint, in a bath of equivalent particles, giving rise to collisions, and eventually to systems equilibration, uniquely towards the Maxwell–Boltzmann distribution

$$W(v) = (2\pi k_B T/m)^{-1/2} e^{-mv^2/(2k_B T)} \tag{1}$$

of velocities.

* Corresponding author. Tel.: +45-353-25507; fax: +45-353-89157.
E-mail addresses: metz@nordita.dk (R. Metzler), theo.nonnenmacher@physik.uni-ulm.de (T.F. Nonnenmacher).

0301-0104/02/$ - see front matter
PII: S0301-0104(02)00537-2

Similarly, diffusion processes occur when there is a gradient in the distribution of a certain substance in another substance. One can distinguish between (self)diffusion of a bulk of molecules of one sort in a bulk of molecules with different physical properties (isotopes, etc.) and Brownian motion, the erratic motion of a larger molecule in a bath of smaller molecules which continuously bombard the larger one such that the latter experiences a net motion. Diffusion as a physical process of everyday experience is mostly a phenomenon associated with the gaseous phase, such as the mixing of transparent air and brownish bromine gas or the spreading of perfume in air. In liquids, diffusion is mostly too slow to be easily perceptible such as the diffusion of sugar in a cup of tea which occurs on the time scale of days. However, on small scales like in biological cells, Brownian motion dominates [3].

The first mathematical description of diffusion is due to Fick [4] whose "second law"

$$\frac{\partial C}{\partial t} = K \frac{\partial^2}{\partial x^2} C(x,t) \tag{2}$$

for the concentration $C(x,t)$ of the substance of interest, which was based on Fourier's 1822 law of heat conduction. Here, K denotes the diffusion constant which is, in essence, a measure for the efficiency of the spreading of the underlying substance.

The breakthrough in understanding diffusion processes came with the connection to random walks and, ultimately, probability theory. Originally described as the "battling" of (dust) particles seen against the sunlight in dark hallways of houses by Roman poet-philosopher Titus Lucretius Carus [5], re-discovered by Dutch physicist-physician Jan Ingenhousz [6] as the jittery motion of fine charcoal dust on an alcohol surface and later by Scottish botanist Robert Brown [7] as zig-zag motion of small pollen grains, the physical nature of this process was investigated further both experimentally and theoretically [8]. It was Albert Einstein [9] in his kinetic approach to diffusion who connected the diffusion equation with random walks, from a probabilistic viewpoint. Accordingly, diffusion is governed by the linear parabolic partial differential equation

$$\frac{\partial W}{\partial t} = K \frac{\partial^2}{\partial x^2} W(x,t) \tag{3}$$

for the probability density function (pdf) $W(x,t)$. The associated distribution $W(x,t)\,\mathrm{d}x$ is the probability to find the random walker at position $x,\ldots,x+\mathrm{d}x$ at time t. This is, the concentration picture becomes a probabilistic approach, and Eqs. (2) and (3) can be related through the normalisation $W(x,t) = C(x,t)/\int C(x,t)\,\mathrm{d}x$. An important finding of Einstein was the relation (now referred to as Einstein relation)

$$K = k_{\mathrm{B}}T/(m\eta) \tag{4}$$

between the diffusion constant K and the friction constant η experienced by the Brownian particle as an effective interaction with the bath particles. This relation was used by Perrin to determine the Avogadro/Loschmidt number by determination of K, through recordings as shown in Fig. 1. Perrin's work was a formidable step in the connection between microscopic physics and macroscopic quantities known from thermodynamics. Perrin's experiments were later refined by Westgren and Kappler [11,12]. It should be noted that actually some of Einstein's results were found previously by Bachelier [13] in his thesis on stock market fluctuations.

The Green's function of the diffusion equation (i.e., the solution for the δ initial condition $W_0(x) = \delta(x)$) is given by the Gaussian

$$W(x,t) = (4\pi K t)^{-1/2} \mathrm{e}^{-x^2/(4Kt)}, \tag{5}$$

which, in turn, produces the mean squared displacement

$$\langle x^2(t) \rangle = 2Kt \tag{6}$$

with the typical linear time-dependence. The universality of the Gaussian solution (5) is guaranteed by the central limit theorem which states that the normalised sum $N^{-1/2}\sum_1^N X_n$ of the independent random vari-

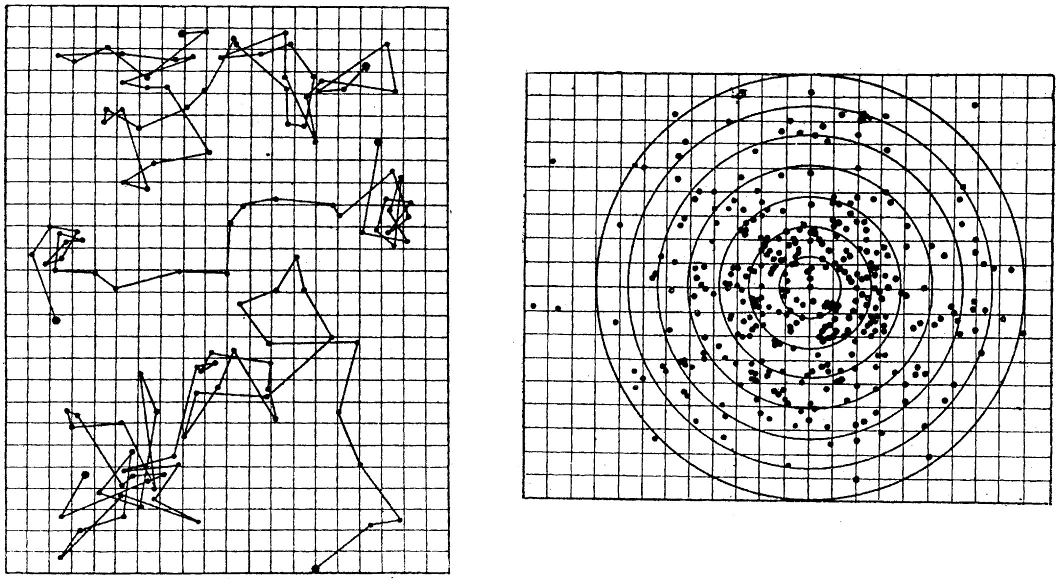

Fig. 1. Random walk trajectories recorded by Perrin [10]. Left: three designs obtained by tracing a small grain of putty at intervals of 30 s. Right: the starting point of each motion event is shifted to the origin. The figure illustrates the continuum approach of the jump length distribution if only a large number of jumps is considered.

able X_n with finite variance in the limit of large sample size N will be Gaussian-distributed, irrespectively of the details of the distribution of the X_n [14,15].

In a broad variety of systems, however, it has been found that correlations in space or time give birth to *anomalous* transport whose pdf is non-Gaussian and/or whose mean squared displacement is non-linear in time or even diverges [16–18]. These systems include charge carrier transport in amorphous semiconductors [19], aqueous solutions of gelatin [20], conducting carbon-black composites [21], tracer dispersion in convection rolls and rotating flows [22], capillary surface waves [23], Richardson pair dispersion [24], solar surface flow [25], the motion of bacteria and the flight of an albatross [26], microrheology in complex fluids such as solutions of polyethylene oxide and actin [27], intracellular transport [28], statistics in econophysics [29], transport in micelles [30], 2-D dusty plasmas [31], the dynamics in (bio)polymeric systems [32], the NMR diffusometry in porous glasses and percolation clusters [33], laser cooling in quantum optical systems [34], and subsurface and catchment transport of chemicals in aquifers [35], among a zoo of others.

In what follows, we introduce some common statistical concepts for the description of normal transport processes and explore their possible extension to anomalous transport processes. We then consider a bifractional diffusion equation which combines Lévy jump statistics with a slowly decaying memory such that x and t decouple. We present analytical solutions for the different regimes which are analysed numerically. These results are put in a perspective with the systems equilibration in the presence of an external field, and with Lévy walks which couple space and time. Finally, we summarise and give a short outlook.

2. Brownian diffusion concepts

In this section we briefly list some approaches to the description of Brownian diffusion. In the next section, we will then see how generalisations of some of these concepts give rise to anomalous diffusion.

(i) *Langevin equation approach.* The physical picture of a molecule which experiences ongoing bombardments through bath molecules is embedded in the Langevin equation [15,36,37]

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \Gamma(t)/\eta, \tag{7}$$

which describes the changes of the position due to the random force $\Gamma(t)$ in time, mediated through the friction constant η. The random force is additive, i.e., independent of x, and such that its mean $\overline{\Gamma(t)} = 0$ vanishes. Brownian motion corresponds to white Gaussian noise in the sense that the distribution of Γ, $p(\Gamma)$, is Gaussian, i.e., the characteristic function of p is of the form e^{-ak^2}; and to the δ-correlation

$$\overline{\Gamma(t)\Gamma(t')} = \Delta\delta(t - t'). \tag{8}$$

Here, the "noise strength" Δ fulfils $\Delta = 2\eta k_\mathrm{B} T/m$ due to the fluctuation–dissipation relation. Actually, Eq. (7) corresponds to the high-friction limit of the phase-space Langevin equation [36]

$$\frac{\mathrm{d}}{\mathrm{d}t}x = v, \quad \frac{\mathrm{d}}{\mathrm{d}t}v = -\eta v + F(x)/m + \Gamma(t). \tag{7a}$$

Eq. (7) can be integrated immediately, and by realising that $\overline{x^2(t)} = x_0^2 + \eta^{-2}\int_0^t \int_0^{t'} \overline{\Gamma(t)\Gamma(t')}\,\mathrm{d}t\,\mathrm{d}t'$, one finds with (8) the second moment (6) and the Einstein relation (4) [15,16,36].

(ii) *Chapman–Kolmogoroff approach.* Combining the Langevin equation (7a) with the fundamental Chapman–Kolmogoroff equation from probability theory [38], one obtains the Klein–Kramers equation, a bivariate Fokker–Planck equation in phase space whose low- and high-friction limits are the Rayleigh and Fokker–Planck–Smoluchowski equations [39,40].

(iii) *Random walk approach.* In the simplest case, consider a 1-D lattice random walk with lattice spacing a. If at a given location at time t, the walker must have just performed a jump from either the left or the right. If only next-neighbour jumps are considered (being of the length of the lattice constant a), the probability $W(x_n, t)$ to find the walker at site $x_n = x_{n\pm1} \mp a$ at time t is defined through

$$W(x_n, t) = \frac{1}{2}W(x_{n-1}, t - \Delta t) + \frac{1}{2}W(x_{n+1}, t - \Delta t). \tag{9}$$

This is actually the simplest form of a master equation [15]. In the continuum limit which has to be taken such that $a^2/\Delta t$ remains finite, we obtain exactly the diffusion equation (3) in $\mathrm{O}(\Delta t)$ and $\mathrm{O}(a^2)$, and with $K \equiv a^2/(2\Delta t)$.

(iv) *Master equation approach.* The master equation [41] is the differential form of the Chapman–Kolmogoroff equation [15]. A special form of a master equation, Eq. (9), shows its intimate connection to random walk processes. Its general form is continuous and allows for distributions of jump lengths [15,16,36].

(v) *Continuity equation plus constitutive law.* The more phenomenological approach to diffusion combines the continuity equation

$$\frac{\partial W}{\partial t} = -\frac{\partial}{\partial x}S(x, t) \tag{10}$$

with the constitutive equation ("Fick's first law")

$$S(x, t) = -K\frac{\partial}{\partial x}W(x, t) \tag{11}$$

to produce Eq. (3). Here, the probability current S is defined such that it drives the diffusing substance to destroy the probability current [4,16].

(vi) *Hamiltonian approach.* This is a valuable approach to diffusion systems in the weak noise limit underlined by Fogedby [42]. Let us consider the simplest case, given by the Langevin equation (7) with noise correlation (8). With the ansatz $W \propto \mathrm{e}^{-S/\Lambda}$, one finds, in the weak noise limit, the Hamilton–Jacobi-type equation

$$\frac{\partial}{\partial t}S + H(x, \partial S/\partial x) = 0$$

for S.

By identifying the canonical momentum $p = \partial S/\partial x$ and energy $E = H$, the "Hamiltonian" of the system is given by $H = p^2/2$, and one can define the "action"

$$S(x,t) = \int \left(p\frac{dx}{dt'} - H \right) dt' \tag{12}$$

such that the principle of least action defines the canonical equations

$$\frac{dx}{dt} = Kp, \quad \frac{dp}{dt} = 0. \tag{13}$$

Generalisations to multidimensional, forced, and even non-linear systems are possible, and therefore the dynamics can be investigated within the language of dynamical systems which allows for a detailed stability analysis [42].

3. Pathways to anomalous diffusion

Let us now consider generalisations of these principles to cases of anomalous diffusion.

(i) *Continuous time random walk (CTRW) approach.* Possibly the most common approach to anomalous diffusion with constant or vanishing external force is the generalised random walk framework, the CTRW introduced by Montroll, and coworkers [43]. Following Klafter et al. [44], we consider random walks which follow a multiple trapping scenario such that the walker gets occasionally trapped at some location. It is immobilised for some waiting time t after which it is released and continues its random walk. Eventually, it will be trapped again and so forth. Individual waiting times are distributed with the waiting time pdf $\psi(\mathrm{t})$. In continuous space, the length $\mathfrak{x}$ of each jump might be additionally distributed according to another pdf, the jump length pdf $\lambda(\mathfrak{x})$. In Fourier–Laplace space defined through

$$W(k,u) \equiv \int_0^\infty \int_{-\infty}^\infty e^{-ut-ikx} W(x,t)\, dx\, dt,$$

it can be shown that the pdf $W(x,t)$ is given by

$$W(k,u) = \frac{1-\psi(u)}{u}\frac{1}{1-\Psi(k,u)}, \tag{14}$$

where $\Psi(\mathfrak{x},\mathrm{t})$ is the jump distribution. If ψ and λ are decoupled, $\Psi(\mathfrak{x},\mathrm{t}) = \psi(\mathrm{t})\lambda(\mathfrak{x})$; the coupled case will be discussed below. In the usual diffusion limit $k \to 0$ and $u \to 0$, Eq. (14) can then be solved for the pdf W. For instance, the Gaussian (5) is recovered for all those ψ and λ which possess at least the first or second moments, respectively, such as $\lambda(\mathfrak{x}) = (4\pi\ell^2)^{-1/2}e^{-\mathfrak{x}^2/(4\ell^2)}$ and $\psi(\mathrm{t}) = \tau e^{-\mathrm{t}/\tau}$ which correspond to $\lambda(k) \sim 1 - \ell^2k^2$ and $\psi(u) \sim 1 - \tau u$. For all forms of $\psi(\mathrm{t})$ and $\lambda(\mathfrak{x})$ which possess diverging moments of first or second order, the associated diffusion process is no longer Gaussian [16,44]. In particular, a waiting time pdf $\psi(\mathrm{t})$ with diverging first moment corresponds to a process with a slowly decaying memory due to the fact that extremely long waiting times t are allowed with a fairly high probability; [1] similarly, a jump length

[1] Note that is is sufficient to to have a $\psi(\mathrm{t})$ with diverging characteristic waiting time to produce a non-Gaussian pdf $W(x,t)$, even for a $\lambda(\mathfrak{x})$ with existing second moment. Conversely, a process with existing characteristic waiting time but diverging step length variance is still Markoffian [16].

pdf $\lambda(\mathfrak{x})$ with diverging second moment allows for extremely long jumps, a process called a Lévy flight [16,17]. These processes are discussed in more detail at hand of the bi-fractional diffusion equation in the following section.

(ii) *Generalised master equation approach.* CTRWalks correspond exactly to a generalised master equation [45], this is, a master equation with a memory kernel

$$W(x,t) = \int_{\infty}^{\infty} \mathrm{d}x' \int_{0}^{t} \mathrm{d}t' \mathfrak{K}(x,x',t-t') W(x',t'), \tag{15}$$

in which the kernel $\mathfrak{K}$ determines the jump length and the waiting times. Actually, in the form (15), also arbitrary external force fields can be incorporated [46].

(iii) *Langevin equation approach.* Fogedby [47] shows that on parametrising the random walk with its arc length s along the trajectory, the position of the random walker is $x(s) = \int_0^s \mathfrak{x}(s')\,\mathrm{d}s'$ where $\mathfrak{x}(s')$ denotes the length of the "sth" step. This corresponds to the Langevin equation

$$\frac{\mathrm{d}x}{\mathrm{d}s} = \mathfrak{x}(s), \tag{16}$$

where the arc length s takes on the role of time which, in turn, is connected to the actual clock time t through the analogous Langevin equation

$$\frac{\mathrm{d}t}{\mathrm{d}s} = \mathfrak{t}(s) \tag{17}$$

involving the waiting times $\mathfrak{t}$, so that the coupled Eqs. (16) and (17) constitute an alternative formulation of CTRWs. It is shown in [47] how certain solution types can be obtained from this formalism.

For a pure Lévy flight, one can directly infer the corresponding space-fractional diffusion and Fokker–Planck equation from the Langevin equation with Lévy noise [48], compare [49].

(iv) *Generalised Chapman–Kolmogoroff equation approach.* In [50], it was demonstrated how a continuous time generalised version of the Chapman–Kolmogoroff equation gives rise to different regimes of anomalous diffusion. For instance, to obtain the subdiffusive domain, the associated multiple trapping process is viewed as trapping periods interrupting a dynamics defined in terms of the standard Langevin equation. Due to its fundamental nature, the generalised Chapman–Kolmogorov equation contains a large pool of stochastic processes from which the types discussed herein can be distilled under certain assumptions [50].

(v) *Characteristic functional approach.* Vlad et al. [51] employ a characteristic functional approach combined with the Huber complex relaxation model to study Lévy-type processes in the one- and, interestingly, many-body picture. A special case of this approach is the space-fractional Fokker–Planck equation discussed in the following.

(vi) *Generalised Hamiltonian approach?* Due to the temporally and spatially non-local nature of generalised diffusion processes of the type discussed herein, a direct generalisation of the above sketched Hamiltonian approach is not feasible. It might be an interesting point for future work to find some generalised least action principle associated with memory or Lévy flight processes.

Trajectories vs the pdf $W(x,t)$. The pdf $W(x,t)$ is an averaged quantity which controls the probability to find the random walking test particle at some position x at time t, after a large number of steps. In contrast, a given *trajectory*, the spatial route followed by the random walker in a particular sequence of steps (like the Perrin images in Fig. 1), keeps track of the microscopic motion of the particle. The trajectory contains less information than the x–t diagram of the kinetics (in which waiting times correspond to horizontal "stagnation lines"), as it is independent of the time spent on a given segment of the path. It is therefore entirely dominated by the jump length distribution λ. In Fig. 2, we compare the two types of trajectories created by a λ with and without a finite variance. It is prominent how singular long jumps separate the trajectory into clusters in Lévy flights. These, when magnified, show an equivalent structure, this statistical self-similarity

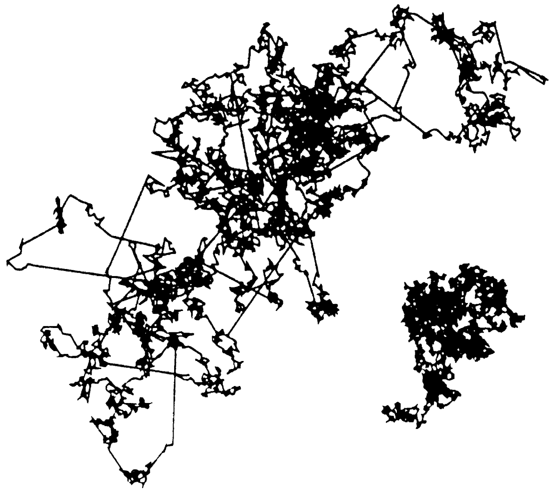

Fig. 2. Lévy trajectory with Lévy index 1.5 consisting of 7000 steps. The lines connect successive locations of the random walker, illustrating the clustering nature which gives rise to the fractal graph dimension (compare [16]). The small trajectory corresponds to a Gaussian random walk with the same number of steps. The space-filling character in this 2-D case contrasts the fractal structure of the Lévy trajectory.

giving rise to a fractal graph dimension of the Lévy trajectory [16]. The connection between the generalised central limit theorem whose limit distributions are Lévy stable distributions are reviewed in the detailed report by Bouchaud and Georges [17].

4. Bi-fractional diffusion and wave equations

Consider a diffusion process which combines long-tailed waiting time and Lévy flight properties, i.e., a decoupled CTRW process with waiting time and jump length pdfs which do not possess a first and second moment, respectively. If we assume that both $\mathfrak{t}$ and $\mathfrak{x}$ are identically distributed random variables whose distributions fall into the basin of the generalised central limit theorem [16,17,52] it follows that the characteristic functions of ψ and λ are given by

$$\psi(u) \equiv \int_0^\infty \psi(\mathfrak{t})\mathrm{e}^{-u\mathfrak{t}}\ \mathrm{d}\mathfrak{t} = \mathrm{e}^{-\tau^\alpha u^\alpha} \tag{18}$$

with $0 < \alpha \leqslant 1$ and [2]

$$\lambda(k) \equiv \int_{-\infty}^\infty \lambda(\mathfrak{x})\mathrm{e}^{ik\mathfrak{x}}\ \mathrm{d}\mathfrak{x} = \mathrm{e}^{-\sigma^\mu|k|^\mu} \tag{19}$$

with $0 < \mu \leqslant 2$, respectively. Eq. (18) defines a one-sided Lévy distribution which corresponds to the asymptotic behaviour $\psi(\mathfrak{t}) \sim (\mathfrak{t}/\tau)^{-1-\alpha}/\tau$ for $0 < \alpha < 1$, and therefore to a diverging characteristic waiting time

$$\mathfrak{T} \equiv \int_0^\infty \mathfrak{t}\psi(\mathfrak{t})\,\mathrm{d}\mathfrak{t}. \tag{20}$$

[2] We only consider symmetric forms for λ. The more general skewed case is discussed in [53].

In the limit $\alpha \to 1$, Eq. (18) reduces to the limiting exponential $\psi(u) = e^{-u\tau}$, and therefore to the δ-form $\psi(t) = \delta(t - \tau)$ of the waiting time pdf, and the characteristic waiting time $\mathfrak{T} = \tau$. In this limit, Markoffian dynamics is recovered. Similarly, Eq. (19) defines a symmetric Lévy distribution which gives rise to the asymptotic power-law behaviour $\lambda(\mathfrak{x}) \sim (|\mathfrak{x}|/\sigma)^{-1-\mu}/\sigma$ $(0 < \mu < 2)$ such that the jump length variance

$$\mathfrak{X}^2 \equiv \int_{-\infty}^{\infty} \mathfrak{x}^2 \lambda(\mathfrak{x})\, d\mathfrak{x} \tag{21}$$

diverges. For $\mu \leqslant 1$, even the corresponding first moment $|\mathfrak{X}| \equiv \int_{-\infty}^{\infty} |\mathfrak{x}|\lambda(\mathfrak{x})\, d\mathfrak{x}$ diverges. In the limit $\mu = 2$, Eq. (19) defines the Gaussian jump length distribution $\lambda(\mathfrak{x}) = (4\pi\sigma^2)^{-1/2} \exp(-\mathfrak{x}^2/(4\sigma^2))$ with variance $\mathfrak{X}^2 = 2\sigma^2$.

The jump pdf $\Psi(\mathfrak{x}, t) = \lambda(\mathfrak{x})\psi(t)$ in the diffusion limit therefore takes on the asymptotic form

$$\Psi(k, u) = 3D\psi(u)\lambda(k) \sim 1 - (u\tau)^{\alpha} - (\sigma|k|)^{\mu}. \tag{22}$$

With relation (14), we obtain the Fourier–Laplace transform

$$W(k, u) = \frac{1/u}{1 + K_{\alpha}^{\mu} u^{-\alpha}|k|^{\mu}} \tag{23}$$

of the propagator $W(x, t)$ where we defined $K_{\alpha}^{\mu} \equiv \sigma^{\mu}/\tau^{\alpha}$.

Multiplying with the denominator of the right-hand side of Eq. (23), we get

$$W(k, u) - \frac{1}{u} = -K_{\alpha}^{\mu} u^{-\alpha}|k|^{\mu} W(k, u). \tag{24}$$

Noting that the Fourier transform of the initial condition $W_0(x) = \delta(x)$ is 1, this equation immediately yields the bi-fractional equation

$$W(x, t) - W_0(x) = K_{\alpha}^{\mu}\, {}_0D_t^{-\alpha} \mathfrak{D}_x^{\mu} W(x, t) \tag{25}$$

in the so-called *integral form* (for the fractional time-integral on the right-hand side), see definitions (30) and (33) below. Applying an ordinary differentiation in time, we arrive at the differential form of Eq. (25), the bi-fractional diffusion equation

$$\frac{\partial W}{\partial t} = K_{\alpha}^{\mu}\, {}_0D_t^{1-\alpha} \mathfrak{D}_x^{\mu} W(x, t) \tag{26}$$

for $0 < \alpha < 1$ and $0 < \mu \leqslant 2$. The generalised diffusion constant bears dimension $[K_{\alpha}^{\mu}] = \mathrm{cm}^{\mu}\ \mathrm{s}^{-\alpha}$. We will discuss Eq. (26) and its solution in more detail in the following section.[3] The mean squared displacement corresponding to Eqs. (25) and (26) is obtained to follow the power-law form

$$\langle x^2(t) \rangle = 2K_{\alpha}^{2} \frac{t^{\alpha}}{\Gamma(1+\alpha)} \tag{27}$$

for $\mu = 2$; otherwise, it *diverges*.[4]

Formally, the bi-fractional diffusion equation can be extended into the parameter range $1 < \alpha < 2$ (see below for more details on restrictions). In this regime, the bi-fractional wave equation [55][5, 6]

$$\frac{\partial^2 W}{\partial t^2} = K_{\alpha}^{\mu}\, {}_0D_t^{2-\alpha} \mathfrak{D}_x^{\mu} W(x, t) \tag{28}$$

[3] Note that Eqs. (25) and (26) can also be written in the decoupled form ${}_0D_t^{\alpha} W(x, t) - \frac{t^{-\alpha} W_0(x)}{\Gamma(1-\alpha)} = K_{\alpha}^{\mu} \frac{\partial^2}{\partial x^2} W(x, t)$. By the explicit occurrence of the initial value term $W_0(x) = W(x, 0)$, the normalisation is preserved, as ${}_0D_t^{\alpha} 1 = t^{-\alpha}/\Gamma(1-\alpha)$ [54].

[4] I.e., the properties of $\mathfrak{T}$ and $\mathfrak{X}^2$ translate to the macroscopic level of the process, signifying the scale-invariant nature of the long power-law tails.

[5] The space-fractional operator enters analogously to the above discussion.

[6] Note that we chose a different notation from [55] for matter of consistency.

with the generalised "velocity" yields. Thus, in the limit $\alpha = 2$ and $\mu = 2$, Eq. (28) reduces to the standard wave equation

$$\frac{\partial^2 W}{\partial t^2} = c^2 \frac{\partial^2}{\partial x^2} W(x,t) \tag{29}$$

with $c^2 \equiv K_2^2$, which describes ballistic propagation with $\langle x^2(t)\rangle = c^2 t^2$. The fractional wave equation can be obtained as an overdamped, force-free limit of two complementary fractional Klein–Kramers models discussed to some extent below. Here, we want to concentrate on its properties.

In Eqs. (25), (26) and (28), the fractional Riemann–Liouville operator is defined through the fractional integral [54]

$${}_0D_t^{-\alpha} W(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^t \mathrm{d}t' \frac{W(x,t')}{(t-t')^{1-\alpha}}, \quad \alpha \in \mathbb{R}^+ \tag{30}$$

combined with the ordinary differential operator such that

$${}_0D_t^{n-\alpha} W(x,t) = \frac{\partial^n}{\partial t^n} \left({}_0D_t^{-\alpha} W(x,t) \right), \quad n \geqslant \alpha \wedge n \in \mathbb{N}. \tag{31}$$

The Riemann–Liouville fractional integral fulfils the important property

$$\int_0^\infty \mathrm{e}^{-ut} \left({}_0D_t^{-\alpha} f(t) \right) \mathrm{d}t = u^{-\alpha} f(u) \tag{32}$$

under Laplace transformation by which from (24) we identified the fractional time derivative in Eq. (25). Conversely, the spatiofractal operator $\mathfrak{D}_x^\mu$ is defined in terms of the Fourier transform

$$\int_{-\infty}^\infty \mathfrak{D}_x^\mu W(x,t) \mathrm{e}^{\mathrm{i}kx} \, \mathrm{d}x = -|k|^\mu W(k,t). \tag{33}$$

$\mathfrak{D}_x^\mu$ is sometimes called fractional Riesz operator, and in one dimension it is, up to the argument, equivalent to the Weyl fractional operator [54].

We note that both generalised equations (26) and (28) reduce to the standard diffusion equation (3) in the limit $\alpha = 1$ and $\mu = 2$, as can be seen from the limit forms of the fractional operators. Both Eq. (25) and the integral form of Eq. (28) in the limit $\mu = 2$ correspond to the fractional diffusion equation introduced by Schneider and Wyss [56]. The combined bi-fractional form was considered by Saichev and Zaslavsky [57]. The space-fractional case was derived by Seshadri and West [58]. The time-fractional wave equation was originally derived by West et al. [59]. Its properties were studied in [55] where the counter-moving two-hump character of the solution was stressed. More recently, the combined space- and time-fractional form was put forward by West and Nonnenmacher [60], and it was investigated by Mainardi et al. [53] and by Luchko and Gorenflo [61]. As the latter references investigate the mathematical properties of Eqs. (26) and (28) in seminal fashion, we restrict our discussion to some more fundamental properties, and prefer to concentrate on the physical embedding and implications of these model equations. Moreover, some new special cases are given particular consideration. In this sense, our treatise complements the recent study by Barkai who focuses on the differences between the CTRW solution and the solution of the fractional model if one departs from the $k \to 0$ and $u \to 0$ limits usually taken to derive fractional equations [62].

5. Solution of the bi-fractional diffusion and wave equations

In this section, we discuss the solution of Eqs. (26) and (28). We concentrate at first on the slow range $0 < \alpha < 1$, and then continue to the fast range $1 < \alpha < 2$. Some interesting special cases and the calculation of moments follow.

5.1. Slow case $0 < \alpha \leqslant 1$ and arbitrary $0 < \mu \leqslant 2$

Rewriting Eq. (23) in the form

$$W(k,u) = \left(u + Ku^{1-\alpha}|k|^{\mu}\right)^{-1}, \tag{34}$$

we recognise the definition of the Mittag-Leffler function in Laplace space [63] which immediately leads to

$$W(k,t) = E_{\alpha}(-K|k|^{\mu}t^{\alpha}). \tag{35}$$

This form describes the mode relaxation of the bi-fractional equations (26) and (28) for a fixed Fourier mode k, and therefore generalises the exponential mode relaxation in the Brownian limit $\alpha = 1$. We note that (35) is symmetric in k, as it should. The Mittag-Leffler function E_{α} has the series expansion [63]

$$E_{\alpha}(-At^{\alpha}) = \sum_{n=0}^{\infty} \frac{(-At^{\alpha})^{n}}{\Gamma(1+\alpha n)} \tag{36}$$

and is the natural generalisation of the exponential function which is contained in (36) in the limit $\alpha = 1$. In the range $0 < \alpha < 1$, the Mittag-Leffler function possesses also the $At^{\alpha} \gg 1$ expansion [63]

$$E_{\alpha}(-At^{\alpha}) = \sum_{n=0}^{\infty} \frac{(-At^{\alpha})^{-1-n}}{\Gamma(1-\alpha-\alpha n)}. \tag{37}$$

In this $0 < \alpha < 1$ domain, it consequently interpolates between the initial stretched exponential law

$$\exp\left(-\frac{At^{\alpha}}{\Gamma(1+\alpha)}\right) \tag{38}$$

and the terminal inverse power-law form

$$E_{\alpha}(-At^{\alpha}) \sim \frac{1}{At^{\alpha}\Gamma(1-\alpha)}. \tag{39}$$

In particular, the mode relaxation (35) is strictly monotonic.

To obtain the propagator in (x,t) space, one could Fourier- and Laplace-invert the Taylor series of expression (23) term by term. Alternatively, one can use the identification

$$W(k,u) = \frac{1}{u} H^{1,1}_{1,1}\left[K^{\mu}_{\alpha}u^{-\alpha}|k|^{\mu} \,\middle|\, \begin{matrix}(0,1)\\(0,1)\end{matrix}\right] \tag{40}$$

with the Fox function $H^{1,1}_{1,1}$ (the general definition of Fox functions is given in Appendix A) and employ the theorems for Fourier- and Laplace-inversion [64]. Here, we already have the (k,t)-representation in terms of the Mittag-Leffler function. The latter is connected to the Fox function through Eq. (A.13). Starting off from

$$W(k,t) = H^{1,1}_{1,2}\left[K^{\mu}_{\alpha}t^{\alpha}|k|^{\mu} \,\middle|\, \begin{matrix}(0,1)\\(0,1),(0,\alpha)\end{matrix}\right], \tag{41}$$

we perform the Fourier-inversion $(2\pi)^{-1}\int_{-\infty}^{\infty} f(k)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}k = \pi^{-1}\int_0^{\infty} f(k)\cos(kx)\,\mathrm{d}k$ where the reduction to the cosine transform is possible due to the point symmetry of $W(k,t)$. Employing the rule (A.12), we find the exact representation

$$W(x,t) = \frac{1}{\mu|x|\sqrt{\pi}} H^{2,1}_{2,3}\left[\frac{|x|}{(2K^{\mu}_{\alpha}t^{\alpha})^{1/\mu}} \,\middle|\, \begin{matrix}\left(1,\frac{1}{\mu}\right),\left(1,\frac{\alpha}{\mu}\right)\\ \left(\frac{1}{2},\frac{1}{2}\right),\left(1,\frac{1}{\mu}\right),\left(1,\frac{1}{2}\right)\end{matrix}\right]. \tag{42}$$

Eq. (42) also solves the fractional wave equation (28), and is therefore the most general solution of the space- and time-fractional diffusion/wave equation. In the latter case, certain restrictions prevail, see below. Solution (42) reduces to the special case for $\mu = 2$,

$$W(x,t) = \frac{1}{\sqrt{2\pi K_\alpha^1 t^\alpha}} H_{1,2}^{2,0}\left[\frac{x^2}{2K_\alpha^1 t^\alpha}\middle| \begin{matrix} \left(1-\frac{\alpha}{2},\alpha\right) \\ (0,1),\left(\frac{1}{2},1\right) \end{matrix}\right], \tag{43}$$

which was considered by Schneider and Wyss [56] for both subdiffusion and sub-ballistic superdiffusion. The series expansion of the most general solution (42) reads

$$W(x,t) = \frac{1}{\mu\sqrt{\pi}|x|}\left(2\sum_{n=0}^{\infty}\frac{\Gamma\left(1-\frac{2}{\mu}\left[\frac{1}{2}+n\right]\right)\Gamma\left(\frac{2}{\mu}\left[\frac{1}{2}+n\right]\right)}{n!\Gamma\left(\frac{1}{2}+n\right)\Gamma\left(1-\frac{2\alpha}{\mu}\left[\frac{1}{2}+n\right]\right)}(-1)^n z^{1+2n} + \mu\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{1}{2}-\frac{\mu}{2}[1+n]\right)}{\Gamma\left(\frac{\mu}{2}[n+1]\right)\Gamma(1-\alpha[n+1])}(-1)^n z^{\mu(1+n)}\right) \tag{44}$$

and therefore the initial behaviour is

$$W(x,t) \sim \begin{cases} \frac{4\Gamma(1-1/\mu)\Gamma(1/\mu)}{\pi\mu\Gamma(1-\alpha/\mu)\left(2K_\alpha^\mu t^\alpha\right)^{1/\mu}}, & 1<\mu<2, \\ \frac{\Gamma(1/2-\mu/2)}{\sqrt{\pi}\Gamma(\mu/2)\Gamma(1-\alpha)|x|^{1-\mu}2K_\alpha^\mu t^\alpha}, & 0<\mu<1 \end{cases} \tag{45}$$

depending on whether $\mu \gtrless 1$, for $z = |x|/(2K_\alpha^\mu t^\alpha)^{1/\mu} \ll 1$. Thus, albeit the convergence to a constant in the $\mu = 2$ case for $0 < \alpha < 1$, in the additionally space-fractional case, the pdf diverges at $x = 0$ for $0 < \mu < 1$. The Cauchy propagator $\mu = 1$ is a borderline case which converges to a constant and therefore belongs to the generic behaviour of the $1 < \mu < 2$ regime.

An alternative way to represent the solution, especially for the purpose of numerical evaluation, can be obtained as follows. Regard the solution of the Markoffian ($\alpha = 1$), but spatially fractional Eq. (26) which we denote $W_1(x,t)$. By help of relation (23), it is straightforward to show that this Markoffian solution is connected to the non-Markoffian, $W_\alpha(x,t)$ for arbitrary α, through the scaling relation [65,66]

$$W_\alpha(x,u) = u^{\alpha-1} W_1(x, K_1^\mu u^\alpha / K_\alpha^\mu). \tag{46}$$

This is equivalent to the generalised Laplace transformation

$$W_\alpha(x,u) = u^{\alpha-1}\int_0^\infty e^{-K_1^\mu u^\alpha t/K_\alpha^\mu}\,dt, \tag{47}$$

from which, in turn, the relation

$$W_\alpha(x,t) = \int_0^\infty E(s,t) W_1(x,s)\,ds \tag{48}$$

follows where the kernel $E(s,t)$ is given by the modified one-sided Lévy distribution [66]

$$E(s,t) = \frac{t}{\alpha s} L_\alpha^+\left(\frac{t}{(s^*)^{1/\alpha}}\right) \tag{49}$$

$$= \frac{1}{\alpha s} H_{1,1}^{1,0}\left[\frac{(s^*)^{1/\alpha}}{t}\middle| \begin{matrix}(1,1)\\ \left(1,\frac{1}{\alpha}\right)\end{matrix}\right] \tag{50}$$

$$= \frac{1}{s}\sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma(1-\alpha-\alpha n)n!}\left(\frac{s^*}{t^\alpha}\right)^{1+n}, \tag{51}$$

with $s^* = K_1^\mu s / K_\alpha^\mu$. From the series expansion of the Fox function, special representations can be obtained for given α. For instance, for $\alpha = 1/2$, we find

$$E(s,t) = \frac{1}{\sqrt{\pi t}} \mathrm{e}^{-(s^*)^2/(4t)}. \tag{52}$$

This representation can then be used to obtain the general solution from the purely space-fractional solution

$$W_1(x,t) = \frac{1}{\mu |x| \sqrt{\pi}} H_{1,2}^{1,1}\left[\frac{|x|}{2K_1^\mu t} \middle| \begin{matrix} \left(1, \frac{1}{\mu}\right) \\ \left(\frac{1}{2}, \frac{1}{2}\right), \left(1, \frac{1}{2}\right) \end{matrix} \right]. \tag{53}$$

Expression (53) reduces to the Cauchy propagator

$$W(x,t) = \frac{1}{2\pi K_1^1 t} \frac{1}{1 + x^2/(K_1^1 t)} \tag{54}$$

in the limit $\mu = 1$. From this, we used Eq. (52) to plot Fig. 3. Here, we see the generic behaviour of both Lévy flight and long-tailed-memory systems. The former is expressed in the slow power-law tails, straight lines in the double-logarithmic plot. The latter causes the persistence of the initial condition (that part of the probability density which corresponds to a random walker which has not moved since $t = 0$) which is visible as distinct and sharp cusps.

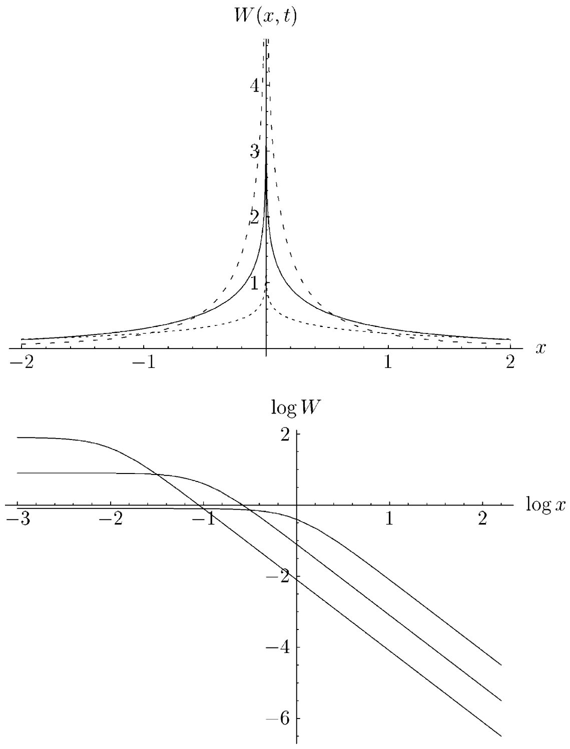

Fig. 3. Propagator $W(x,t)$ of the space- and time-fractional diffusion equation (26) in the Cauchy case $\mu = 1$ and for $\alpha = 1/2$. The dimensionless time steps have been chosen as 0.1, 1, 10. The cusp shape at the origin is reminiscent of the (persistent) initial condition (top). In the $\log_{10}$–$\log_{10}$ representation (bottom), the power-law tails are distinct.

5.2. Enhanced case $1 \leqslant \alpha \leqslant \mu \leqslant 2$

For all combinations of the Lévy indices $0 < \alpha \leqslant 1$ and $0 < \mu \leqslant 2$, the solution of the fractional diffusion equations (26) is a proper pdf, i.e., it is non-negative everywhere. This can be easily proved: take any Lévy distribution, i.e., the solution of (26) for $\alpha = 1$. This is non-negative per se. The transformation (48) is defined in terms of a one-sided Lévy distribution, and from the representation in Laplace space, the positivity of the general solution is obvious.

This is no longer true for all cases in which $\alpha > 1$. For instance, for the combination $\alpha = 3/2$ and $\mu = 1$, one can find the exact representation

$$W(x,t) = \frac{4\sqrt{2K^1_{3/2}t^3}}{3x^2} {}_2F_2\left(1, 1; \frac{5}{6}, \frac{7}{6}; -\frac{4K^1_{3/2}t^3}{27x^2}\right),$$

which becomes negative close to the origin. Instead, one can prove that the inequality [53]

$$1 \leqslant \alpha \leqslant \mu < 2 \tag{55}$$

has to be fulfilled in order to ensure the non-negativity of the general solution (42). This is a non-trivial restriction of the allowed parameter space. Particularly, the first-order moment of the absolute value of the position coordinate, $\langle |x(t)| \rangle$ always exists. From the Fox function representation, the special role of the limiting case $\alpha = \mu$ can already be anticipated as here the essential parameter becomes $M = 0$, according to definition (A.5). Moreover, it was already shown by Schneider and Wyss [56] that the purely time-fractional equation for superdiffusion produces a proper pdf only in one space-dimension ($\mu = 2$).

There exists a similar expression of the enhanced solution in terms of the wave equation solution, as we reported for the case of subdiffusion in Eqs. (46) and (47), see [55] for details.

It is characteristic for the case with $0 < \alpha < 1$ to have sharp cusps at the origin, reminiscent of the persistent initial condition. In the present case $1 < \alpha < 2$, the region around the origin becomes depleted and two counter-moving humps appear. For $\mu = 2$, we display this behaviour in Fig. 4. Note that from Eq. (44) and with the constraint (55), the pdf is finite at $x = 0$.

The Mittag-Leffler representation (35) of the propagator remains valid in the range $1 < \alpha < 2$. However, the behaviour of this function changes. Instead of the completely monotonic decay, stretched exponential fashion turning over to inverse power-law form, oscillations occur. It was shown by Mainardi and Gorenflo [67] that the Mittag-Leffler function $E_\alpha(-\lambda t^\alpha)$ for $\lambda \in \mathbb{R}^+$ can be decomposed into the sum

$$E_\alpha(-\tau^\alpha) = f_\alpha(\tau) + g_\alpha(\tau), \tag{56}$$

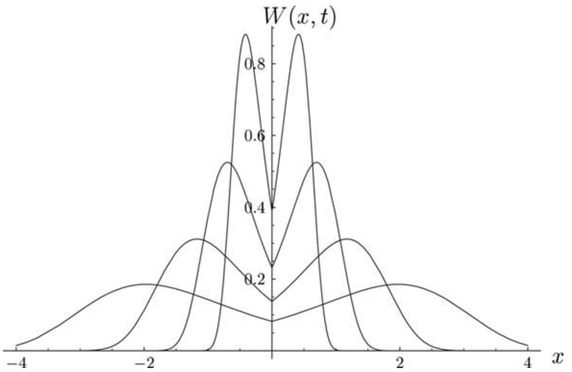

Fig. 4. Superdiffusion with Gaussian jump length distribution, for $\alpha = 3/2$. The antipersistence of the initial condition mirrors the countermoving humps. The wings of the pdf decay exponentially fast. The qualitative behaviour in this linear plot is hardly distinguishable from the neutral-fractional case displayed in Fig. 6, except for the finiteness at the origin.

where we employ the rescaled time $\tau \equiv \lambda^{1/\alpha} t$. Here, f_α decays completely monotonically, whereas g_α contains oscillations. In particular, one has [67]

$$f_\alpha(\tau) = \sum_{n=1}^{N} \frac{(-1)^{n-1} \tau^{-n\alpha}}{\Gamma(1-n\alpha)} + \mathrm{O}(\tau^{-(N+1)\alpha}), \quad \tau \to \infty \tag{57}$$

and

$$g_\alpha(\tau) = \frac{2}{\alpha} \mathrm{e}^{-|\cos(\pi/\alpha)|\tau} \cos\left(\tau \sin\left(\frac{\pi}{\alpha}\right)\right) \tag{58}$$

such that the oscillations die out exponentially fast, and f_α approaches 0 in power-law fashion, from below.

The occurrence of oscillations in the characteristic function of the propagator, $W(k,t)$, stems from the countermotion of the humps. In fact, for the wave equation (29), the travelling wave solution $\frac{1}{2}(\delta(x - \sqrt{K_2^1}t) + \delta(x + \sqrt{K_2^1}t))$ gives rise to the purely oscillatory characteristic function $W(k,t) = \cos(-\sqrt{K_2^1 k^2}t)$.

5.3. The neutral-fractional case

A case of special interest corresponds to the choice $\alpha = \mu$ which belongs to the neutral-fractional class defined in [53]. For this choice, we can simplify expression (42) via the duplication rule (A.14), to obtain

$$W(x,t) = \frac{1}{\mu|x|} H_{2,2}^{1,1}\left[\frac{2^{1/2}|x|}{(K_\mu^\mu)^{1/\mu} t}\left|\begin{matrix}\left(1,\frac{1}{\mu}\right),\left(1,\frac{1}{2}\right)\\ \left(1,\frac{1}{\mu}\right),\left(1,\frac{1}{2}\right)\end{matrix}\right.\right], \tag{59}$$

which has the series expansion

$$W(x,t) = \frac{1}{|x|} \sum_{n=0}^{\infty} \frac{(-1)}{\Gamma\left(\frac{\mu}{2}[1+n]\right)\Gamma\left(1-\frac{\mu}{2}[1+n]\right)} \left(\frac{2^{1-1/\mu}|x|}{(K_\mu^\mu)^{1/\mu} t}\right)^{(1/\mu)(1+n)}. \tag{60}$$

The solution (59) is a borderline case of an H-function as the parameter $M = 0$, compare Eq. (A.5). To obtain the asymptotic behaviour for large argument, we can use theorem (A.6) to invert the argument and derive the expansion

$$W(x,t) \sim \frac{1}{|x|} \sum_{n=1}^{\infty} \frac{(-1)^n}{\Gamma\left(1+\frac{\mu}{2}n\right)\Gamma\left(-\frac{\mu}{2}n\right)} \times \left(\frac{2^{1-1/\mu}|x|}{(K_\mu^\mu)^{1/\mu} t}\right)^{\frac{1}{\mu}(1+n)} \tag{61}$$

$$\sim \frac{K_\mu^\mu t^\mu}{2^{\mu/2}|x|^{1+\mu}} + \cdots, \tag{62}$$

in particular, the typical Lévy-type inverse power law behaviour of index $-1-\mu$ is reproduced.

Let us compare the three different neutral-fractional regimes:

(i) Firstly, the simplest neutral-fractional case is the Cauchy propagator (54) for the parameter combination $\alpha = \mu = 1$. We note in particular that this propagator converges to the value $(2\pi K_1^1 t)^{-1}$ at $x = 0$.

(ii) Secondly, consider the example $\alpha = \mu = 1/2$. In this special case, we find that the H-function representation can be reduced to the simple form

$$W(x,t) = \frac{2}{|x|} \frac{z^{1/2}}{\sqrt{2}\pi + 2\pi z^{1/2} + \sqrt{2}\pi z} \quad \therefore z = \frac{|x|}{2\left(K_{1/2}^{1/2}\right)^2 t}. \tag{63}$$

In Fig. 5, we show the typical behaviour of this solution. The two regimes for small and large argument can be seen in the asymptotic behaviour

$$W(x,t) \sim \begin{cases} \frac{1}{\pi K_{1/2}^{1/2} t^{1/2} |x|^{1/2}}, & z \ll 1, \\ \frac{2K_{1/2}^{1/2} t^{1/2}}{\pi |x|^{3/2}}, & z \gg 1 \end{cases} \tag{64}$$

obtained from Eq. (63), according to which the pdf diverges at the origin as $\sim |x|^{-1/2}$; for large $|x|$ it falls off like $\sim |x|^{-3/2}$.

(iii) This behaviour can be compared with the complementary case $\alpha = \mu = 3/2$, for which we obtain the equally simple form

$$W(x,t) = \frac{\sqrt{2}}{3\pi|x|} \frac{z^{3/2} + z^3 + z^{9/2}}{1 + z^6} \quad \therefore z = \frac{2^{1/3}|x|}{\left(K_{3/2}^{3/2}\right)^{2/3} t}. \tag{65}$$

From the propagator expression (65), we infer the asymptotic behaviour

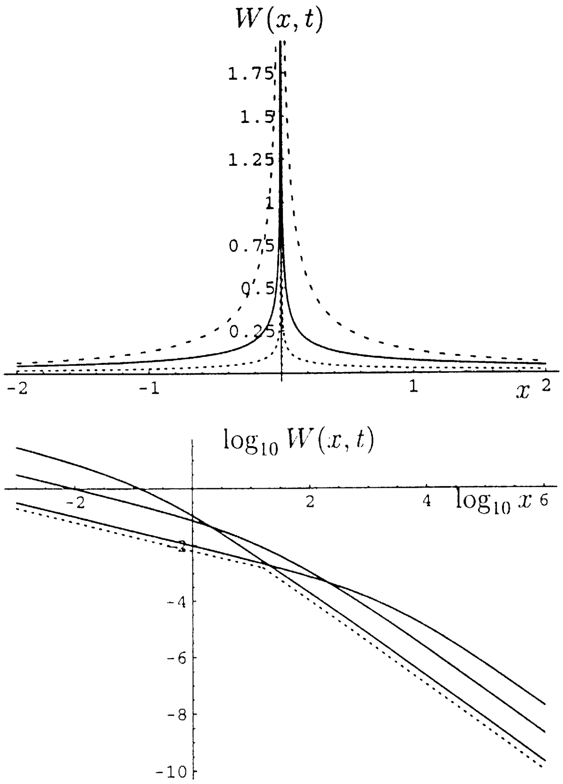

Fig. 5. Anomalous diffusion for the neutral-fractional case $\alpha = \mu = 1/2$. Top: Linear axes, bottom: double-logarithmic scale. Dimensionless times, top: 0.2, 1, 100; bottom: 0.1, 10, 1000. The dashed lines in the bottom plot indicate the slopes −1/2 and −3/2. Note the divergence at the origin.

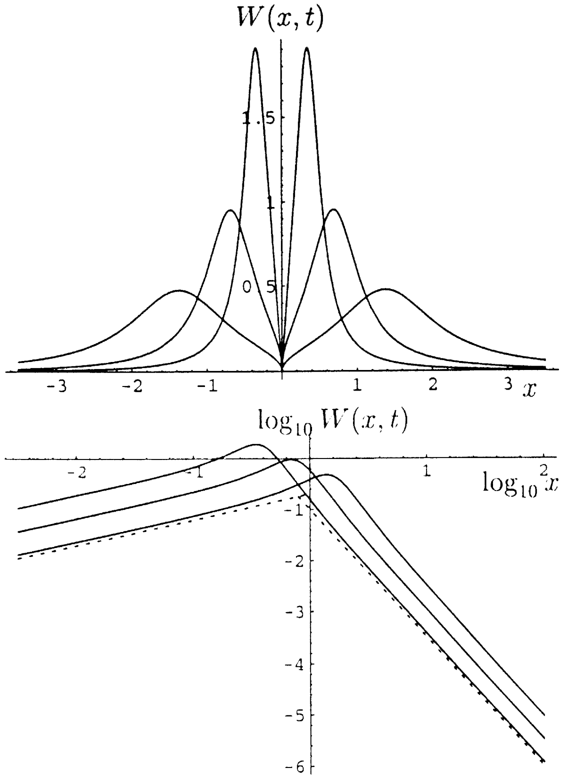

Fig. 6. Anomalous diffusion for the neutral-fractional case $\alpha = \mu = 3/2$. Top: Linear axes, bottom: double-logarithmic scale. Both plots are drawn for the dimensionless times 0.5, 1 and 2. The dashed lines in the bottom plot indicate the slopes 1/2 and −5/2. Note the complete depletion at the origin.

$$W(x,t) \sim \begin{cases} \frac{4\sqrt{2}|x|^{1/2}}{3\pi K_{3/2}^{3/2} t^{3/2}}, & z \ll 1, \\ \frac{\sqrt{2}K_{3/2}^{3/2} t^{3/2}}{12\pi |x|^{1+3/2}}, & z \gg 1. \end{cases} \tag{66}$$

This case is plotted in Fig. 6. Here, we find the opposite behaviour: *complete depletion* at $x = 0$.

Thus, the neutral-fractional case enforces a remarkable behaviour on the (Pólya) returning probability, i.e., the probability to be at the origin at $t > 0$ after preparation of the system: in the subdiffusive case $0 < \alpha = \mu < 1$, the returning probability $W(0,t)$ diverges, whereas in the superdiffusive case $1 < \alpha = \mu < 2$, it vanishes identical to zero. Only the Cauchy borderline case $\alpha = \mu = 1$ converges to a typical returning behaviour, $W(0,t) = 1/(2\pi K_1^1 t)$.

5.4. Fractional order moments

For $0 < \mu < 2$, the mean squared displacement $\langle x^2(t)\rangle$ of the solution (42) diverges. However, it is possible to calculate fractional order moments defined through

$$\langle |x(t)|^{\delta}\rangle = \int_{-\infty}^{\infty} |x|^{\delta} W(x,t)\, dx. \tag{67}$$

Using the definition of the Mellin transform

$$f(s) = \int_0^\infty t^{s-1} f(t)\,\mathrm{d}t, \tag{68}$$

we find with the property (A.3) that

$$\left\langle |x|^\delta(t) \right\rangle = \frac{2^{1+\delta/\mu}}{\mu\pi^{1/2}} (K_\alpha^\mu t^\alpha)^{\delta/\mu} \frac{\Gamma\left(\frac{1}{2}+\frac{\delta}{2}\right)\Gamma\left(1+\frac{\delta}{\mu}\right)\Gamma\left(-\frac{\delta}{\mu}\right)}{\Gamma\left(-\frac{\delta}{2}\right)\Gamma\left(1+\frac{\alpha\delta}{\mu}\right)}. \tag{69}$$

This expression is positive as necessarily $\delta < \mu$.

Special cases included in the result (69) contain the normalisation

$$\lim_{\delta\to 0} \langle |x|^\delta(t) \rangle = 1, \tag{70}$$

anomalous diffusion with regular space derivative,

$$\lim_{\mu,\delta\to 2} \langle |x|^\delta(t) \rangle = \frac{2K_\alpha^2 t^\alpha}{\Gamma(1+\alpha)}, \tag{71}$$

and the neutral-fractional case

$$\lim_{\alpha\to\mu} \langle |x|^\delta(t) \rangle = \frac{2^{1-\delta+\delta/\mu} \sin\left(\frac{\pi}{2}(2+\delta)\right)}{\mu \sin\left(\frac{\pi}{\mu}(\delta+\mu)\right)} \left(K_\mu^\mu t\right)^\delta. \tag{72}$$

6. Fractional Fokker–Planck equations

In the preceding two sections, we have dealt with the combination of space- and time-fractional generalisations of the diffusion equation. Now, we proceed to consider the same kind of process which is subject to an external force. The corresponding space- and time-fractional Fokker–Planck equation in the subdiffusive domain $0 < \alpha \leqslant 1$ reads

$$\frac{\partial W}{\partial t} = {}_0D_t^{1-\alpha} \left(\frac{\partial}{\partial x} \frac{V'(x)}{m\eta_\alpha} + \mathfrak{D}_x^\mu K_\alpha^\mu \right) W(x,t), \tag{73}$$

which was derived in [68] and further discussed in [46]. Here, η_α is the generalised friction coefficient and $V(x) = -\int^x F(x')\,\mathrm{d}x'$ is the external potential which gives rise to the force $F(x)$. In the purely time-fractional case, Eq. (73) was proposed in [65] and later derived in [50]. The purely space-fractional case was originally derived by Fogedby [48] and studied by Jespersen et al. [49].

Consider first the temporal behaviour imposed by Eq. (73) which can be extracted through the method of separation of variables [36]. Introducing the product ansatz $W(x,t) = T(t)\varphi(x)$, the two eigenequations

$$\frac{\mathrm{d}T_n}{\mathrm{d}t} = -\lambda_n\, {}_0D_t^{1-\alpha} T_n(t), \tag{74}$$

$$\left(\frac{\partial}{\partial x} \frac{V'(x)}{m\eta_\alpha} + \mathfrak{D}_x^\mu K_\alpha^\mu \right) \varphi_n(x) = -\lambda_n \varphi(x) \tag{75}$$

yield. The temporal eigenequation (74) corresponds to the fractional relaxation equation discussed in [64] which is solved by the Mittag-Leffler function

$$T_n(t) = E_\alpha(-\lambda_n t^\alpha) \tag{76}$$

and therefore corresponds to the mode relaxation of the fractional diffusion equation. The spatial eigenequation (75) for $\mu = 2$ is identical to the one of the standard Fokker–Planck equation. For non-pathological external potentials, the eigenvalues can be arranged such that $0 \leqslant \lambda_1 < \lambda_2, \ldots$, and for a non-trivial force the lowest eigenvalue is $\lambda_0 = 0$ such that a stationary solution exists.

For the $\mu = 2$ case, the stationary solution $W_{\rm st}(x) \equiv \lim_{t\to\infty} W(x,t)$ is given by the exponential form

$$W_{\rm st}(x) = N e^{-V'(x)/(m\eta_\alpha K_\alpha^2)} \tag{77}$$

from which, by comparison with the Gibbs–Boltzmann distribution, the generalised Einstein relation

$$K_\alpha^2 = \frac{k_B T}{m\eta_\alpha} \tag{78}$$

can be inferred. Moreover, in the presence of the constant force field F_0, the first moment $\langle x(t)\rangle_{F_0}$ is related to the second moment $\langle x^2(t)\rangle_0$ in the absence of a force through the linear response relation [65]

$$\langle x(t)\rangle_{F_0} = \frac{F_0}{2}\frac{\langle x^2(t)\rangle_0}{k_B T}. \tag{79}$$

These are typical Gibbs–Boltzmann equilibrium properties connected to the fluctuation–dissipation theorem and linear response.

In the Lévy case $0 < \mu < 2$, a stationary solution for non-trivial external fields does exist; however, this stationary solution decays inverse power-law fashion such that for external potentials which fall off at most like the harmonic Ornstein–Uhlenbeck potential the stationary solution has the Lévy index μ [49], and for any steeper potential $V \propto |x|^{-\beta}$, it falls off with the index β [69]. In particular, it is therefore far off standard Gibbs–Boltzmann equilibrium, and no immediate generalisation of the Einstein–Stokes or the linear response relations can be found. It remains open what the exact meaning of such stationarity solutions from a thermodynamics point of view is. It should be mentioned that the generalised q-statistics approach (Tsallis entropy) [70] leads to non-linear generalisations of the Fokker–Planck equation, and therefore to a different kind of stochastic process.

In the enhanced diffusive case corresponding to the fractional wave equation (28), two approaches have been reported in [71,72], which base on generalisations of the Klein–Kramers equation. Both reduce to the fractional wave equation (28) in the force-free limit. They are based on a Drude-like picture or a collision model, respectively, i.e., the test particle moves with a certain velocity for times governed by a long-tailed $\psi(t)$. Both models predict equilibration of the velocity distribution towards the Maxwell–Boltzmann distribution (1). Roughly speaking, however, the force is supposed to act constantly in [71] whereas it corresponds to "point-like" interactions in [72]. It is interesting to note that the model in [72] leads to equilibration towards the Gibbs–Boltzmann distribution in position space, whereas [71] predicts that there is no stationary solution in x-space. This might correspond to the ever-spreading cutoffs in the tails of the pdf, as found for the Lévy walk approach mentioned below. In fact, it is claimed in [71] that the underlying equation produces lower-order moments of Lévy walks in an external potential.

7. Lévy walk-type description

The above transport models were based on the diffusion limit (in essence, the $k \to 0$, $u \to 0$ limit) of the decoupled form $\Psi(x,t)$ of the jump pdf. It is an essential consequence of this model that for Lévy-type jump length statistics, the mean squared displacement $\langle x^2(t)\rangle$ diverges. An alternative formulation of CTRWs is to consider coupled forms of the jump pdf, i.e., [44,45,73,74]

$$\Psi(x,t) = \psi(t)p_1(x|t) = \lambda(x)p_2(t|x). \tag{80}$$

This is, for either a given jump length pdf $\lambda(\mathfrak{x})$, the corresponding waiting time pdf becomes a conditional probability, $\psi(t)p_1(\mathfrak{x}|t)$, or vise versa. In essence, the introduction of the coupling enforces a time cost to an individual jump: long jumps are penalised by a larger time cost. This statement becomes more transparent by looking at the most prominent example for coupled jump pdfs, the δ-coupling [44]

$$\Psi(\mathfrak{x},t) = C|\mathfrak{x}|^{-\kappa}\delta\big(|\mathfrak{x}| - vt^{\beta}\big). \tag{81}$$

We could thus call the decoupled case a kinematic formulation and Lévy walks a dynamic formulation of anomalous random walks. (In the last sentence, we included the word anomalous as for Brownian random walks there is no difference between the two approaches.) If there is no external field, Lévy walks within the CTRW formalism give rise to a finite mean squared displacement [44] and a pdf which successively approaches a Lévy stable distribution, its wings however cut off by two spikes which correspond to particles moving at maximum velocity (ballistic motion for $\beta = 1$) [75]. It is a matter of current investigations how Lévy walks in extended fields can be formulated in terms of dynamic equations.

8. Conclusions

We have summarised a number of approaches to the stochastic behaviour immanent to Brownian motion and explored possible generalisations to anomalous transport cases. Such Strange kinetics is characteristic for a large, and growing, number of systems which exhibit some form of disorder, and they are typically of power-law form. This, in turn, suggests that the stochastic-dynamical processes in these systems are governed by the generalised central limit theorem, predicting Lévy stable forms for either the jump length pdf or the waiting time pdf, or both. In the present work, we have dealt with such kinds of systems which decouple in space and time.

In these asymptotic power-law cases, fractional equations can be formulated on the level of deterministic equations (i.e., on the noise-averaged level), as a direct and natural generalisation to the standard deterministic equations such as the diffusion equation. The immediateness of this generalisation is especially obvious in the Fourier/Laplace transformed versions. As operator equations, fractional equations can be readily dealt with. Whereas in the force-free case, they are asymptotically equivalent to other formulations like the CTRW, the clear advantage of the fractional approach is seen if descriptions in phase space and in the presence of external force fields are sought. Here, the disadvantage of the other approaches is the question of how to implement a given form of the external force and how to calculate the corresponding moments or the pdf.

The present investigation being mainly based on the force-free limit, is therefore of a more fundamental nature, trying to forge together the different, in a way complementary, effects arising from long-tailed distributions in either time or space. We have considered the fractional diffusion equation with both a Riemann–Liouville fractional operator in time and a Riesz–Weyl fractional operator in space. In this course, two simple and, to our best knowledge, hitherto unknown special cases of neutral-fractional kind were found. For all cases, the Fox function representation can be used to obtain closed form analytical solutions.

It is typical for these systems that they relax according to the Mittag-Leffler pattern which ultimately reaches a slow inverse power-law form such that stationarity is reached by a process with diverging characteristic time scale. Moreover, in cases of Lévy jump length statistics, this equilibrium is of Lévy stable or more steep inverse power-law character. Thus, a still open question concerns the physical meaning of the "equilibrium" of such additive (extensive) processes which ultimately deviate from the exponential Gibbs–Boltzmann form.

It was pointed out by Berry [76] that transport on fractals should be governed by "diffractals", i.e., by properties which involve fractal parameters in both space- and time-evolution. Such diffractals are characterised by a short-wave limit in which ever finer details of similarity structure are explored by the propagating waves. A physical interpretation of this phenomenon is given by Berry [76], and more recently West and Nonnenmacher [60] discussed this problem on the basis of a fractional calculus approach. However, there are numerous open questions to be discussed within the mathematical physics of exotic transport processes with non-existing internal time- and space-scales.

Acknowledgements

We thank Yossi Klafter for helpful discussions and Walter Strunz for bringing the Lucretius citation to our attention. RM acknowledges financial support from the Deutsche Forschungsgemeinschaft (DFG) within the Emmy Noether programme.

Appendix A. The Fox *H*-function

The class of Fox or *H*-functions comprises a large class of special functions known in mathematical physics, such as Meijer's *G*-function, (generalised) Bessel functions, (generalised) hypergeometric functions or the (generalised) Mittag-Leffler functions, just to mention a few. Due to the known series representation given below (and an analogous expression for large argument), special representations can be obtained from the given theorems, or by evaluating these series in symbolic mathematics programs.

The Fox or *H*-function is defined in terms of the Mellin–Barnes type integral [77–80]

$$H^{m,n}_{p,q}\left[z\left|\begin{matrix}(a_1,A_1),(a_2,A_2),\ldots,(a_p,A_p)\\(b_1,B_1),(b_2,B_2),\ldots,(b_q,B_q)\end{matrix}\right.\right]=\frac{1}{2\pi\mathrm{i}}\int_L \chi(s)z^s\,\mathrm{d}s, \tag{A.1}$$

where m, n, p and q are integers satisfying $0\leqslant n\leqslant p$ and $1\leqslant m\leqslant q$, and

$$\chi(s)=\frac{\prod_1^m \Gamma(b_j-B_js)\prod_1^n\Gamma(1-a_j+A_js)}{\prod_{m+1}^q\Gamma(1-b_j+B_js)\prod_{n+1}^p\Gamma(a_j-A_js)}. \tag{A.2}$$

Equivalently, it can be defined by its Mellin transform

$$\int_0^\infty H^{m,n}_{p,q}\left[az\left|\begin{matrix}(a_1,A_1),(a_2,A_2),\ldots,(a_p,A_p)\\(b_1,B_1),(b_2,B_2),\ldots,(b_q,B_q)\end{matrix}\right.\right]z^{s-1}\,\mathrm{d}z=a^{-s}\chi(s). \tag{A.3}$$

Here, the parameters have to be defined such that $A_j>0$ and $B_j>0$ and

$$a_j(b_h+v)\neq B_h(a_j-\lambda-1), \tag{A.4}$$

where $v,\lambda=0,1,2,\ldots$, $h=1,2,\ldots,m$ and $j=1,2,\ldots,m$. L is a contour which separates the poles of $\Gamma(b_j-B_js)$ for $j=1,2,\ldots,m$ from those of $\Gamma(1-a_j+A_js)$ for $j=1,2,\ldots,n$ [79]. The *H*-function is analytic in z if either (i) $z\neq 0$ and $M>0$ or (ii) $0<|z|<B^{-1}$ and $M=0$, where

$$M=\sum_{j=1}^q B_j-\sum_{j=1}^p A_j \quad\text{and}\quad B=\prod_{j=1}^p A_j^{A_j}\prod_{j=1}^q B_j^{-B_j}. \tag{A.5}$$

The *H*-function possesses a number of interesting properties from which we list the ones we employed in our presentation [79]:

(i) If $M=0$, it is sometimes useful to apply the inversion theorem

$$H^{m,n}_{p,q}\left[z\left|\begin{matrix}(a_1,A_1),(a_2,A_2),\ldots,(a_p,A_p)\\(b_1,B_1),(b_2,B_2),\ldots,(b_q,B_q)\end{matrix}\right.\right]=H^{n,m}_{q,p}\left[\frac{1}{z}\left|\begin{matrix}(1-b_1,B_1),(1-b_2,B_2),\ldots,(1-b_q,B_q)\\(1-a_1,A_1),(1-a_2,A_2),\ldots,(1-a_p,A_p)\end{matrix}\right.\right] \tag{A.6}$$

to switch the argument into its inverse.

(ii) If one of the (a_j, A_j) $(j = 1, \ldots, n)$ is equal to one of the (b_j, B_j) $(j = m+1, \ldots, q)$, or likewise one of the (b_j, B_j) $(j = 1, \ldots, m)$ is equal to one of the (a_j, A_j) $(j = n+1, \ldots, p)$, the H-function reduces to one of the lower order according to the following scheme:

$$H^{m,n}_{p,q}\left[z\left|\begin{matrix}(a_1,A_1),(a_2,A_2),\ldots,(a_p,A_p)\\(b_1,B_1),(b_2,B_2),\ldots,(b_{q-1},B_{q-1}),(a_1,A_1)\end{matrix}\right.\right] = H^{m,n-1}_{p-1,q-1}\left[z\left|\begin{matrix}(a_2,A_2),\ldots,(a_p,A_p)\\(b_1,B_1),\ldots,(b_{q-1},B_{q-1})\end{matrix}\right.\right] \quad (A.7)$$

if $n \geqslant 1$ and $q > m$. Note that the parameter pairs are symmetric in the four groups defined through m, n, p and q.

(iii) If $k > 0$,

$$H^{m,n}_{p,q}\left[z\left|\begin{matrix}(a_1,A_1),(a_2,A_2),\ldots,(a_p,A_p)\\(b_1,B_1),(b_2,B_2),\ldots,(b_q,B_q)\end{matrix}\right.\right] = kH^{m,n}_{p,q}\left[z\left|\begin{matrix}(a_1,kA_1),(a_2,kA_2),\ldots,(a_p,kA_p)\\(b_1,kB_1),(b_2,kB_2),\ldots,(b_q,kB_q)\end{matrix}\right.\right]. \quad (A.8)$$

(iv) The multiplication rule reads

$$z^\sigma H^{m,n}_{p,q}\left[z\left|\begin{matrix}(a_1,A_1),(a_2,A_2),\ldots,(a_p,A_p)\\(b_1,B_1),(b_2,B_2),\ldots,(b_q,B_q)\end{matrix}\right.\right]$$
$$= H^{m,n}_{p,q}\left[z\left|\begin{matrix}(a_1+\sigma A_1,A_1),(a_2+\sigma A_2,A_2),\ldots,(a_p+\sigma A_p,A_p)\\(b_1+\sigma B_1,B_1),(b_2+\sigma B_2,B_2),\ldots,(b_q+\sigma B_q,B_q)\end{matrix}\right.\right]. \quad (A.9)$$

(v) If the poles of $\prod_{j=1}^m \Gamma(b_j - B_j s)$ are simple, the following series expansion is valid:

$$H^{m,n}_{p,q}\left[z\left|\begin{matrix}(a_1,A_1),(a_2,A_2),\ldots,(a_p,A_p)\\(b_1,B_1),(b_2,B_2),\ldots,(b_q,B_q)\end{matrix}\right.\right] = \sum_{h=1}^{m}\sum_{v=0}^{\infty}\frac{\prod_{j=1,j\neq h}^{m}\Gamma\left(b_j - B_j\frac{b_h+v}{B_h}\right)\prod_{j=1}^{n}\Gamma\left(1-a_j+A_j\frac{b_h+v}{B_h}\right)}{\prod_{j=m+1}^{q}\Gamma\left(1-b_j+B_j\frac{b_h+v}{B_h}\right)\prod_{n+1}^{p}\Gamma\left(a_j-A_j\frac{b_h+v}{B_h}\right)}\frac{(-1)^v z^{(b_h+v)/B_h}}{v!B_h}. \quad (A.10)$$

(vi) For similar conditions, the asymptotic expansion for large argument ($|z| \to \infty$) holds [78]:

$$H^{m,n}_{p,q}\left[z\left|\begin{matrix}(a_1,A_1),(a_2,A_2),\ldots,(a_p,A_p)\\(b_1,B_1),(b_2,B_2),\ldots,(b_q,B_q)\end{matrix}\right.\right] = \sum_{v=0}^{\infty}\mathrm{res}(\chi(s)z^s) \quad @ \quad s = \frac{a_j - 1 - v}{A_j}, \quad j = 1,2,\ldots,n. \quad (A.11)$$

(vii) Under Fourier cosine transformation, the H-function transforms as

$$\int_0^\infty H^{m,n}_{p,q}\left[x\left|\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right.\right]\cos(kx)\,\mathrm{d}x$$
$$= \frac{\pi}{k}H^{n+1,m}_{q+1,p+2}\left[k\left|\begin{matrix}(1-b_q,B_q),\left(1,\frac{1}{2}\right)\\(1,1),(1-a_p,A_p),\left(1,\frac{1}{2}\right)\end{matrix}\right.\right]. \quad (A.12)$$

(viii) The Mittag-Leffler function is a special case of the H-function

$$E_\alpha(-z) = H^{1,1}_{1,2}\left[z\left|\begin{matrix}(0,1)\\(0,1),(0,\alpha)\end{matrix}\right.\right]. \quad (A.13)$$

Moreover, the duplication rule

$$2\pi^{1/2}\Gamma(2z) = 2^{2z}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) \quad (A.14)$$

of the Γ-function is frequently used to simplify expressions for special H-functions.

References

[1] E. Garber, S.G. Brush, C.W.F. Everitt (Eds.), Maxwell on Molecules and Gases, MIT Press, Cambridge, MA, 1986.
[2] L. Boltzmann, Weitere Studien über das Wärmegleichgewicht unter Gasmolekülen, Sitzber. Österr. Akad. Wiss. (1872); C. Cercignani, Theory and Application of the Boltzmann Equation, Elsevier/Scottish Academic Press, New York/London, 1975.
[3] R. Metzler, Phys. Rev. Lett. 87 (2001) 068103.
[4] A. Fick, Ann. Phys. (Leipzig) 170 (1855) 50.
[5] T.L. Carus, De rerum natura (50 BCE), On the Nature of Things, Harvard University Press, Cambridge, MA, 1975.
[6] J. Ingenhousz, Nouvelles expériences et observations sur divers objets de physique, T. Barrois le jeune, Paris, 1785.
[7] R. Brown, Philos. Mag. 4 (1828) 161; Ann. Phys. Chem. 14 (1828) 294.
[8] M. Gouy, J. Phys. (Paris) 7 (1888) 561; F.M. Exner, Ann. Phys. 2 (1900) 843; K. Pearson, Nature 72 (1905) 294, 342; J.W. Strutt, Lord Rayleigh, Nature 72 (1905) 318; G. Pólya, Math. Ann. 83 (1921) 149.
[9] A. Einstein, Ann. Phys. 17 (1905) 549; Ann. Phys. 19 (1906) 371; R. Fürth (Ed.), Albert Einstein – Investigations on the Theory of the Brownian Movement, Dover, New York, 1956, Einstein actually introduced the name "Brownian motion" although he did not have access to Brown's original work.
[10] J. Perrin, Comptes Rendus (Paris) 146 (1908) 967; Ann. Chim. et d. Phys., VIII 18 (1909) 5.
[11] A. Westgren, Z. Phys. Chem. 83 (1913) 151; Z. Phys. Chem. 89 (1914) 63; Arch. f. Mat. Astr. och Fysik 11 (8&14) (1916); Z. Anorg. Chem. 93 (1915) 231; Z. Anorg. Chem. 95 (1916) 39.
[12] E. Kappler, Ann. Phys. (Leipzig) 11 (1931) 233.
[13] L. Bachelier, Ann. Sci. l'École norm. 17 (1900) 21; comp. P.H. Cootner (Ed.), The Random Character of Stock Market Prices, MIT Press, Cambridge, MA, 1964.
[14] P. Lévy, Processus stochastiques et mouvement Brownien, Gauthier-Villars, Paris, 1965.
[15] N.G. van Kampen, Stochastic Processes in Physics and Chemistry, North-Holland, Amsterdam, 1981.
[16] B.D. Hughes, Random Walks and Random Environments, Vol. 1: Random Walks, Oxford University Press, Oxford, 1995.
[17] J.-P. Bouchaud, A. Georges, Phys. Rep. 195 (1990) 127.
[18] R. Kutner, A. Pękalski, K. Sznajd-Weron (Eds.), Anomalous Diffusion, From Basics to Applications, Springer, Berlin, 1999.
[19] G. Pfister, H. Scher, Adv. Phys. 27 (1978) 747; Q. Gu, E.A. Schiff, S. Grebner, R. Schwartz, Phys. Rev. Lett. 76 (1996) 3196.
[20] S.Z. Ren, W.F. Shi, W.B. Zhang, C.M. Sorensen, Phys. Rev. A 45 (1992) 2416.
[21] L.J. Adriaanse, J.A. Reedijk, P.A.A. Teunissen, H.B. Brom, M.A.J. Michels, J.C.M. Brokken-Zijp, Phys. Rev. Lett. 78 (1997) 1755.
[22] O. Cardoso, P. Tabeling, Europhys. Lett. 7 (1988) 225; T.H. Solomon, E.R. Weeks, H.L. Swinney, Phys. Rev. Lett. 71 (1993) 3975.
[23] A.E. Hansen, E. Schroder, P. Alstrom, J.S. Andersen, M.T. Levinsen, Phys. Rev. Lett. 79 (1997) 1845.
[24] M.-C. Jullien, J. Paret, P. Tabeling, Phys. Rev. Lett. 82 (1999) 2872; I.M. Sokolov, A. Blumen, J. Klafter, Europhys. Lett. 47 (1999) 152.
[25] J.K. Lawrence, A.C. Cadavid, A. Ruzmaikin, T.E. Berker, Phys. Rev. Lett. 86 (2001) 5894.
[26] M. Levandowsky, B.S. White, F.L. Schuster, Acta Protozool. 36 (1997) 237; G.M. Viswanathan, V. Afanasyev, S.V. Buldyrev, E.J. Murphy, P.A. Prince, H.E. Stanley, Nature 381 (1996) 413; Nature 401 (1999) 911.
[27] T.G. Mason, K. Ganesan, J.H. van Zanten, D. Wirtz, S.C. Kuo, Phys. Rev. Lett. 79 (1997) 3282; J. Xu, A. Palmer, D. Wirtz, Macromol. 31 (1998) 6486.
[28] A. Caspi, R. Granek, M. Elbaum, Phys. Rev. Lett. 85 (2000) 5655.
[29] J.F. Muzy, J. Delour, E. Bacry, Euro. Phys. J. B 17 (2000) 537; J.-P. Bouchaud, M. Potters, Theory of Financial Risk: From Statistical Physics to Risk Management, Cambridge University Press, Cambridge, 1999; R. Mantegna, H.E. Stanley, An Introduction to Econophysics, Cambridge University Press, Cambridge, 2000.
[30] A. Ott, J.-P. Bouchaud, D. Langevin, W. Urbach, Phys. Rev. Lett. 65 (1990) 2201.

[31] W.T. Juan, L. I, Phys. Rev. Lett. 80 (1998) 3073.
[32] F. Amblard, A.C. Maggs, B. Yurke, A.N. Pargellis, S. Leibler, Phys. Rev. Lett. 77 (1996) 4470;
R. Kimmich, R.-O. Seitter, U. Beginn, M. Moeller, N. Fatkullin, Chem. Phys. Lett. 307 (1999) 147.
[33] S. Stapf, R. Kimmich, R.-O. Seitter, Phys. Rev. Lett. 75 (1995) 2855;
A. Klemm, R. Metzler, R. Kimmich, Phys. Rev. E 65 (2002) 021112.
[34] S. Schaufler, W.P. Schleich, V.P. Yakovlev, Europhys. Lett. 39 (1997) 383;
Phys. Rev. Lett. 83 (1999) 3162;
Laser Phys. 9 (1999) 277.
[35] R.C. Sidle, B. Nilsson, M. Hansen, J. Fredericia, Water Resour. Res. 34 (1998) 2515, and references therein;
J.M. Boggs, S.C. Young, L.M. Beard, L.W. Gelhar, K.R. Rehfledt, E.E. Adams, Water Resour. Res. 28 (1992) 3281;
E.E. Adams, L.W. Gelhar, Water. Resour. Res. 28 (1992) 3293;
J.C.S. Long, K. Karasaki, Swedish Nuclear Fuel and Waste Management Co., Stockholm, Technical Report No. 92-06, 1992;
B. Berkowitz, H. Scher, Phys. Rev. E 57 (1998) 5858;
J.W. Kirchner, X. Feng, C. Neal, Nature 403 (2000) 524;
B. Berkowitz, R. Metzler, J. Klafter, H. Scher, Water Resour. Res., in press;
R. Metzler, B. Berkowitz, J. Klafter, H. Scher, Geophys. Res. Lett. 29 (2002) 10.1029/2001gL014123;
B. Berkowitz, H. Scher, Phys. Rev. Lett. 79 (1997) 4038.
[36] H. Risken, The Fokker–Planck Equation, Springer, Berlin, 1989.
[37] P. Langevin, Comptes Rendus 146 (1908) 530.
[38] S. Chapman, Proc. R. Soc. (London), Ser. A 119 (1928) 35;
A. Kolmogoroff, Math. Ann. 104 (1931) 415;
Math. Ann. 108 (1933) 149.
[39] S. Chandrasekhar, Rev. Mod. Phys. 15 (1943) 1;
N. Wax (Ed.), Selected Papers on Noise and Stochastic Processes, Dover, New York, 1954, also contains the Chandrasekhar review.
[40] O. Klein, Arkiv Mat. Astr. Fys. 16 (5) (1922);
H.A. Kramers, Physica 7 (1940) 284;
J.W. Strutt, Lord Rayleigh, Philos. Mag. 32 (1891) 424;
A.D. Fokker, Ann. Phys. (Leipzig) 43 (1914) 810;
M. Planck, Sitzber. Preuß. Akad. Wiss. (1917) 324;
M.V. Smoluchowski, Ann. Phys. 48 (1915) 1103.
[41] A. Nordsieck, W.E. Lamb, G.E. Uhlenbeck, Physica 7 (1940) 344.
[42] H.C. Fogedby, Phys. Rev. E 59 (1999) 5065;
Phys. Rev. E 60 (1999) 4950.
[43] E.W. Montroll, J. SIAM 4 (1956) 241;
E.W. Montroll, G.H. Weiss, J. Math. Phys. 10 (1969) 753;
H. Scher, E.W. Montroll, Phys. Rev. B 12 (1975) 2455;
G. Pfister, H. Scher, Phys. Rev. B 15 (1977) 2062;
G.H. Weiss, R.J. Rubin, Adv. Chem. Phys. 52 (1983) 363;
J. Klafter, M.F. Shlesinger, G. Zumofen, Phys. Today 49 (2) (1996) 33;
H. Scher, M.F. Shlesinger, J.T. Bendler, Phys. Today 44 (1991) 26.
[44] J. Klafter, A. Blumen, M.F. Shlesinger, Phys. Rev. A 35 (1987) 3081.
[45] J. Klafter, R. Silbey, Phys. Rev. Lett. 44 (1980) 55.
[46] R. Metzler, Euro. Phys. J. B19 (2001) 249.
[47] H.C. Fogedby, Phys. Rev. E 50 (1994) 1657;
Phys. Rev. E 58 (1998) 1690.
[48] H.C. Fogedby, Phys. Rev. Lett. 73 (1994) 2517.
[49] S. Jespersen, R. Metzler, H.C. Fogedby, Phys. Rev. E 59 (1999) 2736.
[50] R. Metzler, J. Klafter, J. Phys. Chem. B 104 (2000) 3851;
Phys. Rev. E 61 (2000) 6308;
R. Metzler, Phys. Rev. E 62 (2000) 6233.
[51] M.O. Vlad, J. Ross, F.W. Schneider, Phys. Rev. E 62 (2000) 1743.
[52] P. Lévy, Théorie de l'addition des variables aléatoires, Gauthier-Villars, Paris, 1954;
B.V. Gnedenko, A.N. Kolmogorov, Limit Distributions for Sums of Random Variables, Addison-Wesley, Reading, MA, 1954.
[53] F. Mainardi, Y. Luchko, G. Pagnini, Fract. Calc. Appl. Anal. 4 (2001) 153.
[54] K.B. Oldham, J. Spanier, The Fractional Calculus, Academic Press, New York, 1974.
[55] R. Metzler, J. Klafter, Europhys. Lett. 51 (2000) 492.

[56] W.R. Schneider, W. Wyss, J. Math. Phys. 30 (1989) 134.
[57] A.I. Saichev, G.M. Zaslavsky, Chaos 7 (1997) 753.
[58] V. Seshadri, B.J. West, Proc. Natl. Acad. Sci. USA 79 (1982) 4501;
B.J. West, V. Seshadri, Phsyica 113A (1982) 203.
[59] B.J. West, P. Grigolini, R. Metzler, T.F. Nonnenmacher, Phys. Rev. E 55 (1997) 99;
R. Metzler, T.F. Nonnenmacher, Phys. Rev. E 57 (1998) 6409.
[60] B.J. West, T.F. Nonnenmacher, Phys. Lett. A 278 (2001) 255.
[61] Y. Luchko, R. Gorenflo, Fract. Calc. Appl. Anal. 1 (1998) 63.
[62] E. Barkai, Chem. Phys. (2002) this issue.
[63] G.M. Mittag-Leffler, C. R. Acad. Sci. Paris 137 (1903) 554;
R. Acad. dei Lincei, Rendiconti 13 (1904) 3;
Acta Math. 29 (1905) 101;
A. Erdélyi (Ed.), Tables of Integral Transforms (Bateman Manuscript Project, Vol. I), McGraw-Hill, New York, 1954.
[64] W.G. Glöckle, T.F. Nonnenmacher, J. Stat. Phys. 71 (1993) 755.
[65] R. Metzler, E. Barkai, J. Klafter, Phys. Rev. Lett. 82 (1999) 3563.
[66] R. Metzler, J. Klafter, Phys. Rep. 339 (2000) 1.
[67] F. Mainardi, R. Gorenflo, J. Comp. Appl. Math. 118 (2000) 283.
[68] R. Metzler, E. Barkai, J. Klafter, Europhys. Lett. 46 (1999) 431.
[69] A. Chechkin, V. Gonchar, R. Gorenflo, F. Mainardi, L. Tanatarov, cond-mat/0012155; see also A. Chechkin, V. Gonchar, J. Klafter, R. Metzler, L. Tantarov, Chem. Phys. (2002) this issue.
[70] C. Tsallis, S.V.F. Lévy, A.M.C. Souza, R. Maynard, Phys. Rev. Lett. 75 (1995) 3589;
C. Tsallis, D.J. Bukman, Phys. Rev. E 54 (1996) R2197;
L. Borland, Phys. Rev. E 57 (1998) 6634.
[71] R. Metzler, I. Sokolov, Europhys. Lett. 58 (2002) 482.
[72] E. Barkai, R. Silbey, J. Phys. Chem. B 104 (2000) 3866.
[73] M.F. Shlesinger, J. Klafter, Y.M. Wong, J. Stat. Phys. 27 (1982) 499;
M.F. Shlesinger, J. Klafter, Phys. Rev. Lett. 54 (1985) 2551.
[74] J. Klafter, M.F. Shlesinger, G. Zumofen, Phys. Today 49 (2) (1996) 33;
M.F. Shlesinger, G.M. Zaslavsky, J. Klafter, Nature 363 (1993) 31.
[75] J. Klafter, G. Zumofen, J. Phys. Chem. 98 (1994) 7366;
G. Zumofen, J. Klafter, Phys. Rev. E 47 (1993) 851.
[76] M.V. Berry, J. Phys. A 12 (1979) 781.
[77] C. Fox, Trans. Am. Math. Soc. 98 (1961) 395.
[78] B.L.J. Braaksma, Compos. Math. 15 (1964) 239.
[79] A.M. Mathai, R.K. Saxena, The *H*-function with Applications in Statistics and Other Disciplines, Wiley Eastern Ltd., New Delhi, 1978.
[80] H.M. Srivastava, K.C. Gupta, S.P. Goyal, The *H*-functions of one and two variables with applications, South Asian Publishers, New Delhi, 1982.