J. Phys. A: Math. Gen. **31** (1998) 3839–3847. Printed in the UK PII: S0305-4470(98)88018-9

Bi-asymptotic fractals: Fractals between lower and upper bounds

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Received 30 September 1997, in final form 5 January 1998

Abstract. Extending the idea of Rigaut's asymptotic fractals issuing a turnover from a constant to a power-law behaviour towards smaller scales, we extend this idea to asymptotic fractals with both lower *and* upper turnover points, i.e. the fractal region is terminated, towards larger scales, by another constant value. This behaviour is typical for natural fractals, such biological cell boundaries.

Here, we present a new analytic function describing this *bi-asymptotic* behaviour. The introduced parameters can be directly interpreted. We show the advantage of this function in fitting processed images of natural and mathematical fractals, in comparison with standard procedures: the determined dimension is significantly more accurate. A program package with this new function is available.

1. Introduction

Fractals, power laws and Lévy statistics have been commonly used terms in physics, biology, medicine and even engineering for quite a long time. As nature seems to optimize its structures, the existence of self-similarity (i.e. the loss of a length scale) comes into existence quite naturally. Be it fractal coastlines that show much better resistance to a rough sea than any artificial ('civilized') dyke or be it a fractally branched lung issuing the finest possible distribution for incoming oxygen, such systems are much less affected by external distortions than non-fractal pendants would be. Thus, it is not surprising that fractal descriptions have become an important tool in science.

In contrast to (ideal) mathematical fractals such as Koch curves or Cantor sets, nature shows scaling properties only between limits [1]. A typical measurement which may be the result of a box-counting algorithm of a 'natural fractal' is shown in figure 1. There, only region II is governed by a power law and can thus be labelled as a fractal. This peculiar behaviour results from the finite size of all natural objects: for very *low scales* either the measuring device shows a limited resolving power, or the measured object is constituted by certain grains, such as trees in a percolation forest, cells in a tissue or the molecules in a polymer solution. This leads to a *minimum length scale*. All objects also have a *maximum size*, the upper turnover expresses that the measuring length scale is too coarse to see any more details of the object. In the box-counting picture, for example, one box is enough to cover the whole object.

Since Jean-Paul Rigaut communicated his ideas of asymptotic fractals in the early 1980s [6], Benoît Mandelbrot's notion of fractals had to be modified. Thus, natural fractals do

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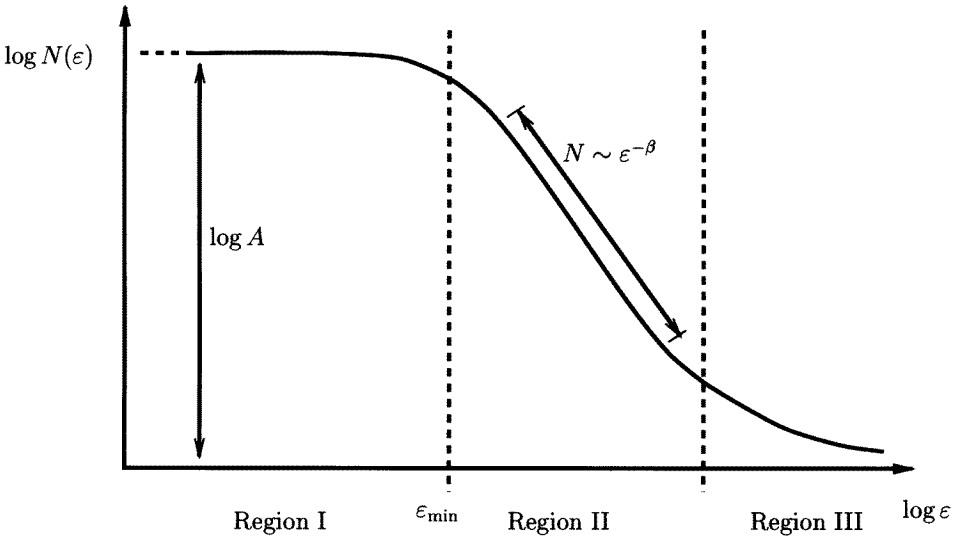

Figure 1. The three typical regions of an asymptotic fractal. Typically, natural fractals are limited by a lower and an upper bound and only show fractality in region II, as indicated by the straight line.

show some smallest length scale terminating the scale-invariance, i.e. the linear behaviour of some quantity in the log–log plot cannot extend to arbitrarily small resolutions. Rigaut was the first to give a formal analytic approach to this phenomenon which he coined *asymptotic fractal*. However, we argued that, for all natural objects, there also exists a largest length scale, i.e. a finite overall size. This caused us to consider fractals issuing turnover points to non-fractal behaviour for both a smallest and a largest length scale; we call them *bi-as* (*bi-asymptotic*) or *inter-fractals*.

In this paper we discuss a new analytic function which modifies an earlier suggestion by Barth [2–4]. The parameters in this approach are not cryptic, they may directly be interpreted as minimum size, fractal dimension etc. We show its application to artificial (Koch curve) and natural (lymphocyte cell boundaries) examples.

At this point we do not hesitate to mention that the analytic structure of our bi-as approach is numerically equivalent to exact solutions of sophisticated relaxation models [5]. These exact solutions cause major numerical problems in fitting. Thus our formula is also an interesting method of finding approximate parameter values for these models.

2. Towards bi-asymptotic fractals

Rigaut [6] was the first to suggest an analytic function taking the lower turnover point into consideration. He called this behaviour asymptotic fractals. His well known formula

$$N(\varepsilon) = \frac{A}{1 + B\varepsilon^n} \tag{1}$$

issues the turnover from a constant behaviour $N \sim A$ to the (asymptotic) fractal power law $N \sim \varepsilon^n$. Here and in the following, $N(\varepsilon)$ denotes the number of boxes (yardsticks) required to cover the underlying object for a given resolution ε, see for example [7, 8]. The length is then simply $L(\varepsilon) = \varepsilon N(\varepsilon)$. However, this description neglects the finite overall size of

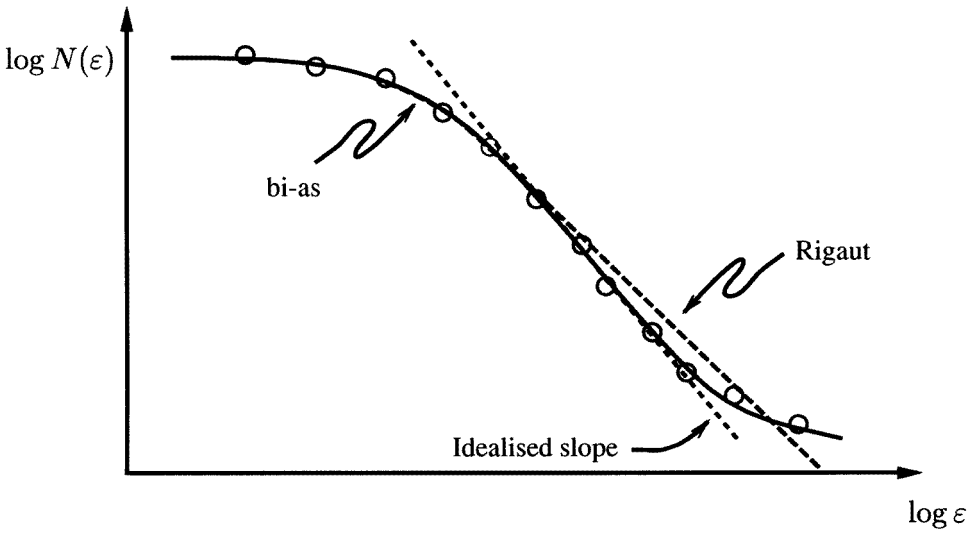

Figure 2. Qualitative comparison of Rigaut-type asymptotic fractal and the bi-as description. The upper turnover causes the fractal dimension extrapolated from Rigaut's formula to be too small. The same is true for a linear fit. The bi-as function follows the turnover. The idealized slope, recovered from the bi-as fit data, exceeds the apparent slope, as discussed in the text.

the object to be measured. In addition, the assumption to obtain the fractal dimension in the very large-scale limit will in general differ from the true dimension for numerical reasons: if there is already influence of the upper turnover point the extrapolated dimension will be too small (see figure 2).

To account for the upper bound of the fractal region, Barth [9] implemented the formula

$$N(\varepsilon) = [(e\varepsilon^{-b} + a\varepsilon^{-d})^{c/b} + 1]^{1/c} \tag{2}$$

which involves the five parameters a to e and goes like ε^{-1} for small values of ε. Formula (2) is a special case of Barth's [2] more general formula

$$\chi^q(\varepsilon) = ((f_1(\varepsilon)^s + f_2(\varepsilon))^{r/s} + f_3(\varepsilon)^r)^{1/r} \tag{3}$$

where χ^q is the sum of the qth power of the measure of each box and scales like $\chi^q(\varepsilon) \sim \varepsilon^{(q-1)D_q}$ in the fractal region. In the case $q = 0$, χ^q is just the number of occupied boxes $\chi^0 = N(\varepsilon)$ [2, 8].

Now we suggest the following formula:

$$N(\varepsilon) = \left[A^{1/\delta} \left(\frac{\varepsilon}{\varepsilon_{\min}} \right)^{-\alpha/\delta} \left(1 + \left(\frac{\varepsilon}{\varepsilon_{\min}} \right)^{\beta/\gamma} \right)^{-\gamma/\delta} + 1 \right]^{\delta} \tag{4}$$

including the five parameters A, $\varepsilon_{\min}$, β, γ and δ. The additional parameter $\alpha \in \{0, 1\}$ is chosen to be zero or one, according to the measuring methods for geometric fractals and how they deal with length scales smaller than a pixel. Box counting usually gives zero-dimensional behaviour† while the yardstick method‡ may give one-dimensional behaviour.

† If you deal with pixels as points (as all programs we have ever seen do) you get zero-dimensional behaviour, but if a pixel is a small square (as one could easily imagine) you get two-dimensional behaviour. Therefore, this point depends crucially on the underlying program.

‡ Yardstick dimension is also known as the divider, compass or ruler dimension.

Of course, α may be taken as an additional parameter describing an arbitrary initial slope in the log–log plot. This may be useful with fractals which change their fractal behaviour with length scale.

The major advantages of this formula are that all parameters are positive, therefore one need not consider many different cases concerning all combinations of sign. The parameters have a well known interpretation, this is important to estimate the start values for a numerical fit.

A is the initial value, if region I is dominated by a flat hill. Otherwise it is the amplitude of the initial power law $\varepsilon^{-\alpha}$. $\varepsilon_{\min}$ describes the turnover point from region I to II. For $\varepsilon \ll \varepsilon_{\min}$, one has either zero or negative slope, according to the choice of α. In the fractal region II the slope is $-(\alpha+\beta)$. In fact $-(\alpha+\beta)$ is an upper limit† for this slope which will be reached in the case of an infinite scaling window. Therefore $-(\alpha+\beta)$ gives a better estimation for our fractal dimension than the slope at any point of the curve. This is very useful if the scaling region is small.

γ determines the width of the curvature at the turnover point I to II, whereas δ is responsible for the turnover curvature II to III. In region III $N(\varepsilon)$ converges to 1. The turnover point is at $\varepsilon_{\max} = A^{\frac{1}{\alpha+\beta}}\varepsilon_{\min}$.

In the following, we turn to applications of the presented formula (4) to natural and mathematical fractals and show the superior properties of this function.

3. Extraction of fractal parameters

In this section we describe the general procedure: how the fractal parameters can be extracted from a given set of data, i.e. a scanned-in picture.

Determining fractal parameters of natural fractals involves four steps.

(1) *Preparation*. Transformation of the object in question to a computer handable format. This may be taking a photograph, using an image-scanner and doing some image processing.

(2) *Analysis*. Determination of the properties in question on different length scales. This may just be the number of occupied boxes when the box-dimension is wanted.

(3) *Data fit*. The measured quantity will depend on length scale with some analytical law. In the ideal fractal case this is a power law and we can fit a straight line in the double-logarithmic plot. The more general case of bi-as fractals is the subject of this paper.

(4) *Presentation of result*. Show the fit in a graph and print out the fractal parameter.

Additionally to our fit routine we developed a shell script which calls the necessary programs to do the last three steps automatically.

Here we restrict ourselves to the determination of fractal dimension, i.e. the measured quantity is the number of occupied boxes or needed yardsticks $N(\varepsilon)$. Of course the fit procedure can be applied to multifractal measures too.

4. The fit routine `<<SimpleFanalfit>>`

`SimpleFanalfit` is part of the fractal analyses package `fanal++` which provides a collection of some fractal analyses programs such as box counting, some image preprocessing such as border extraction and some programs to create example fractals. All this can be used with the help of a script and a Graphical User Interface. The source

† $\alpha+\beta \geqslant |\text{slope}|$ in the double-logarithmic plot.

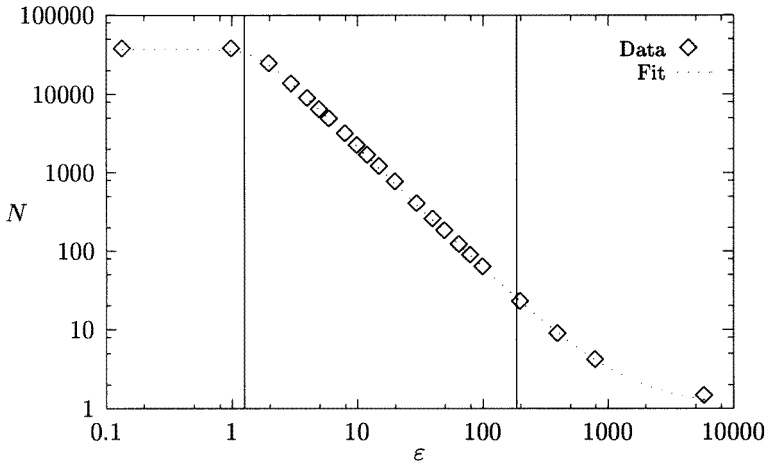

Figure 3. The analyses of the Sierpiński gasket yields the following parameter values: $\alpha = 0.000$, $\beta = 1.574$, $\gamma = 0.388$, $\delta = 1.442$, $A = 3.689 \times 10^4$, $\varepsilon_{\min} = 1.621$. The fractal dimension is 1.574, in comparison with the exact value $\log 3/\log 2 = 1.585$. The plot is produced automatically by our program.

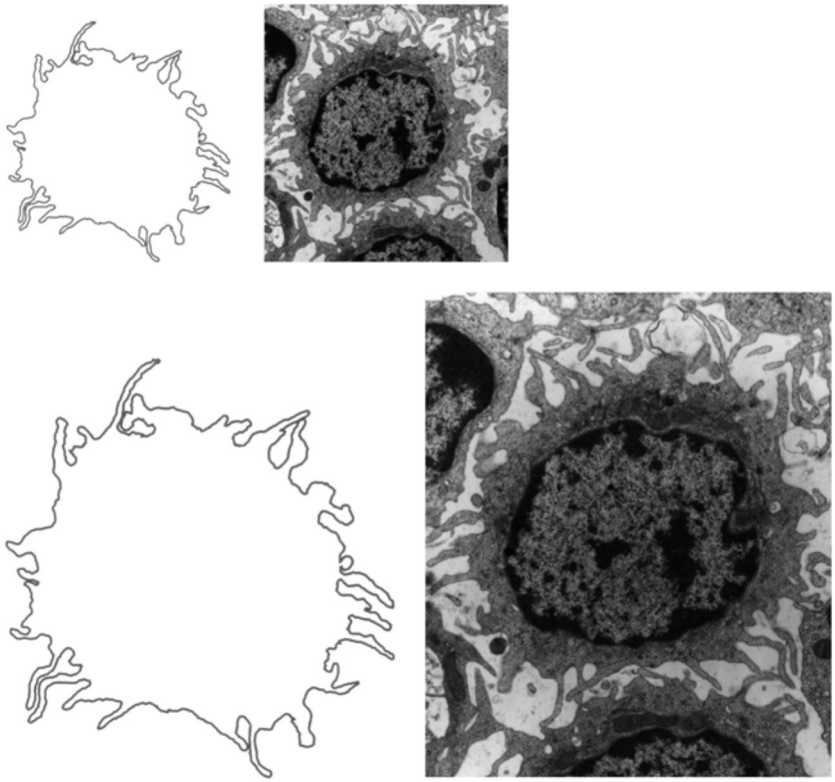

Figure 4. The image of the hairy leukaemic cell at two different resolutions (magnifications are 18 400× and 8700×) and the corresponding extracted cell contours.

code of the program package can be obtained on request from the authors. (The installation will be without problems on `Unix` systems. It should be possible on Windows NT.)

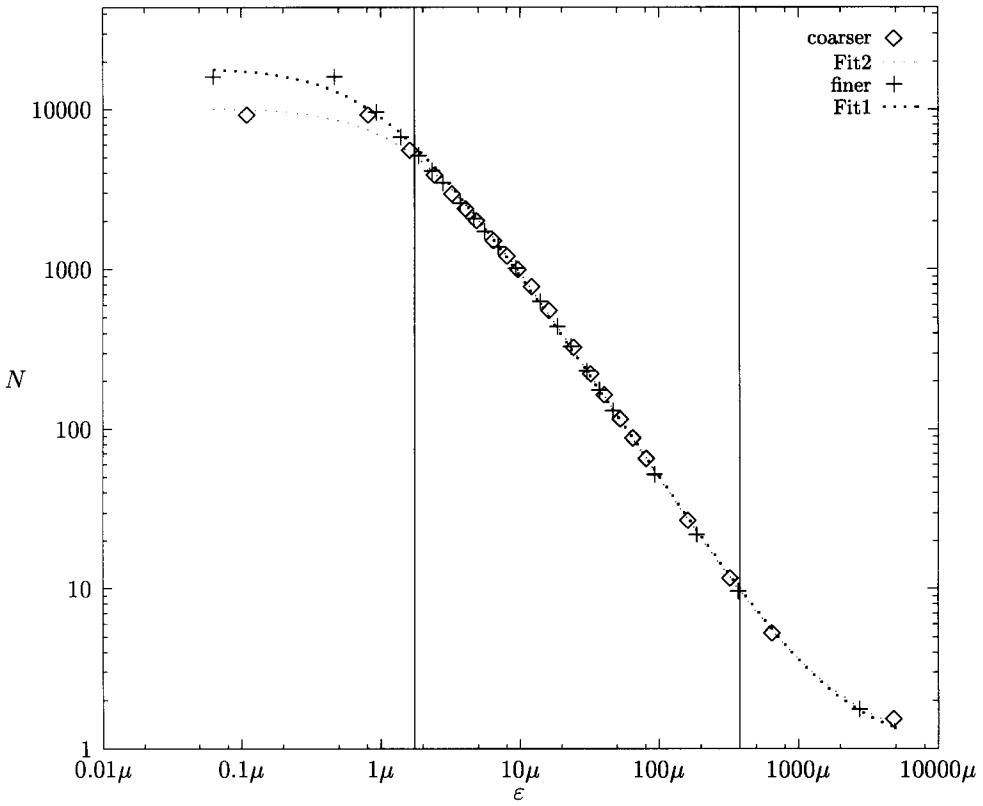

Figure 5. Box counting analyses of a hairy leukaemic cell at two different resolutions. The fits lead to the following parameters for the larger picture: $\alpha = 0.00$, $\beta = 1.29$, $\gamma = 1.04$, $\delta = 1.02$, $A = 1.85 \times 10^4$, $\varepsilon_{\min} = 2.12$. The fractal dimension is 1.29. For the smaller picture: $\alpha = 0.00$, $\beta = 1.34$, $\gamma = 0.86$, $\delta = 1.22$, $A = 1.02 \times 10^4$ $\varepsilon_{\min} = 2.23$. The fractal dimension is 1.34. These results are in perfect agreement with the numbers already reported by Nonnenmacher and Losa [10].

`SimpleFanalfit` is ready for use, in so far as it determines the starting values, necessary for the fit procedure, without further information by the user. The program then determines the slope of the possibly fractal region II, and the other parameters.

We have also included a very important feature for the user. The program calculates the optimum interval, in which real fractal behaviour exists, i.e. where the data can be fitted adequately by a straight line†. The error interval allowed for this approximation, can be preset by the user.

The output is a LaTeX-picture of the double-logarithmic plot with a wide range of possibilities for presentation (see the `man` pages in the program package).

Let us introduce our program with the application to a given fractal, here a Sierpiński gasket (10 generations). Even an exact fractal such as the Sierpiński gasket is a bi-as fractal as soon as it is given as a picture on the computer. The lower limit is the size of a pixel whereas the upper limit is the size of the whole picture. Figure 3 shows the output

† We determine the line through the point of inflection with the computed dimension as slope and its deviation from the fit function.

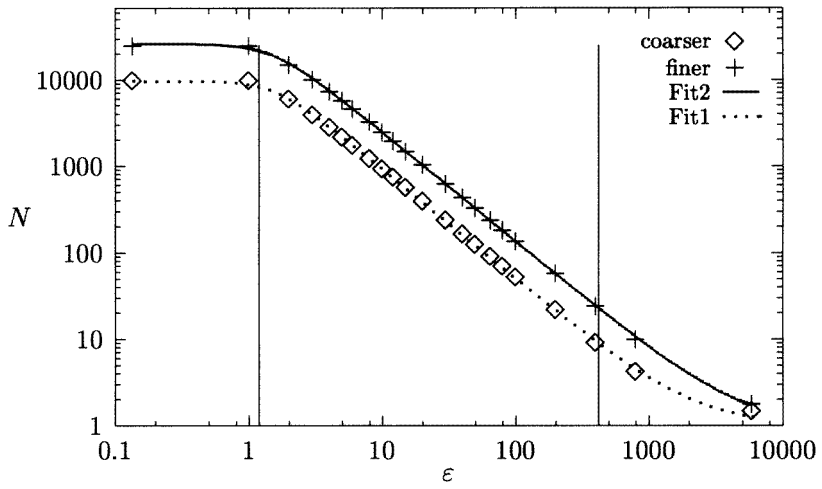

Figure 6. Box-counting analyses of a Koch snowflake at two different resolutions. The fits lead to the following parameters for the larger picture: $\alpha = 0.00$, $\beta = 1.26$, $\gamma = 0.48$, $\delta = 0.94$, $A = 2.63 \times 10^4$, $\varepsilon_{\min} = 1.52$. The fractal dimension is 1.26. For the smaller picture: $\alpha = 0.00$, $\beta = 1.26$, $\gamma = 0.42$, $\delta = 0.97$, $A = 9.77 \times 10^3$, $\varepsilon_{\min} = 1.50$. The fractal dimension is 1.26 matching the exact value $\log 4/\log 3$.

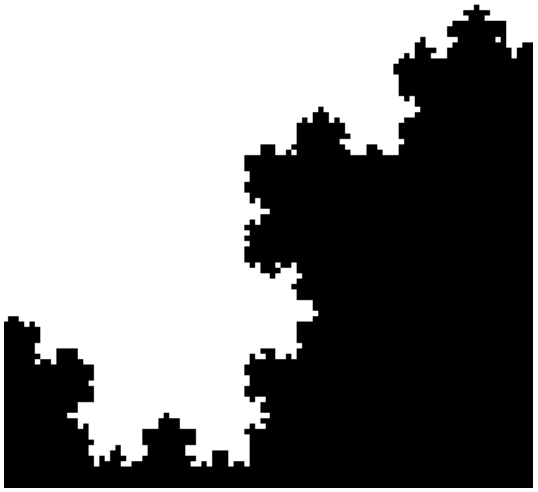

Figure 7. A deep zoom in the image of the von Koch snowflake showing the artefacts produced by scanning and post-processing. The pixels can be seen clearly. The analysis in figure 6 is performed for the contour of the whole snowflake.

of the analyses program. The result of 1.57 comes very close to the exact dimension of $\log 3/\log 2 = 1.58\ldots$.

5. More advanced examples

In order to check the fitness of our formula and the fit procedure we analysed the resolution series of a hairy leukaemic cell (figure 4) which we had already measured earlier [1, 10],

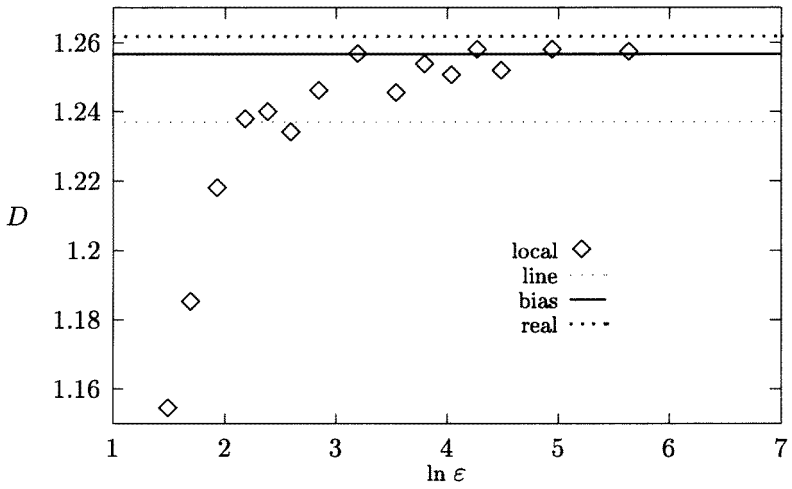

Figure 8. Comparison of the results of different methods in determining the fractal dimension of the von Koch snowflake. The bias method shows a result close to the real value of $\log 4/\log 3 = 1.26\ldots$. The straight line is—as one would expect—too low. Even the local method taking the slope between neighbouring data points shows deficiencies in comparison with the bias method, as the scattering is too large.

and a von Koch snowflake which was subject to the same procedure normally applied to real fractals to study the artefacts (see figure 7) of this procedure.

In the fractal region of the plot of the hairy leukaemic cell (figure 5) the curves coincide because the ε-axis is scaled to the real length of the cell. This is not the case in figure 6 (the von Koch snowflake has no real length scale, we therefore use the pixel as unit).

Normally we would expect the bigger images with finer details to give much better results. But our fit-formula does an excellent job: the dimensions do not differ much more than the expected accuracy. Furthermore our fit formula gives a better result than the usual straight line fit. Figure 8 gives an overview of the different dimensions of the big Koch snowflake: the 'local' dimensions determined by two-point estimates, the result of a linear regression, our bi-as result and the real dimension $\log 4/\log 3$.

6. Conclusions

We presented an analytic function describing bi-as fractals, i.e. fractals issuing an upper and lower turnover point. This function provides a significantly better description of natural fractals than previous concepts such as asymptotic fractals. Especially for numerical purposes, we demonstrated the superiority of the new function.

Many scientists applying fractal concepts are faced with the problem of determining the fractal dimension of bias fractals in a *standardized procedure*. Usual procedures often result in dimensions that are relatively far off from the correct value. The presented analytic formula has the advantage that it can also draw information from the two regions below and above the fractal region. Thus all measured data points can be used. In addition, the other determined parameters give valuable information on the underlying fractal object.

A program package called `fanal++` can be obtained on request from the authors. It includes a user-friendly interface so that scientists, who are not familiar with programming, have easy access to `SimpleFanalfit`.

Acknowledgments

Financial support from the Deutsche Forschungsgemeinschaft, SFB 239, is thankfully acknowledged. Special thanks go to Gabriele Losa who generously provided the pictures of the investigated lymphocytic cells and gave valuable information. We thank Gerd Baumann, Manfred Köpf and Albrecht K Kleinschmidt for discussions.

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