University of Potsdam Institute of Physics and Astronomy
Lecture Stochastic Processes (SS 2018)
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Problem Set 9

(discussion on June 5th)

1. Power spectral density of a particle in a harmonic potential

For a stationary stochastic process $x(t)$ with autocorrelation function $c_x(\tau)$ derive the relations

$$\langle x^*(\omega)x(\omega')\rangle = 2\pi c_x(\omega)\delta(\omega'-\omega) \tag{1}$$

and for $v = dx/dt$

$$c_v(\omega) = \omega^2 c_x(\omega) \tag{2}$$

where an argument ω denotes the Fourier transform. Use these relations to derive the power spectral densities $c_x(\omega)$ and $c_p(\omega)$ for a Brownian particle in a harmonic potential, evolving according to the Langevin equations

$$\frac{d}{dt}x = \frac{p}{m}, \qquad \frac{d}{dt}p = -m\omega^2 x - \zeta\frac{p}{m} + \delta F_p(t). \tag{3}$$

2. Power spectral density and Laplace transform

We consider the Laplace transform

$$\tilde{x}(z) = \int_0^\infty x(t)e^{-zt}dt \tag{4}$$

of a stochastic process with complex argument where $\mathrm{Re}[z] > 0$. Derive the relation

$$c_x(\omega) = \lim_{\varepsilon\to 0}\varepsilon\left\langle\left|\tilde{x}\left(\frac{\varepsilon}{2} - i\omega\right)\right|^2\right\rangle. \tag{5}$$

Find the PSD for stationary shot noise

$$y(t) = \sum_{n=-\infty}^{\infty} x_i\delta(t-t_i) \tag{6}$$

with i.i.d. weights $x_i \sim \lambda(x)$, $\langle x\rangle = 0$ and inter-spike intervals $t_{n+1} - t_n = \tau_i \sim \psi(\tau)$.