University of Potsdam — Institute of Physics and Astronomy
Lecture Stochastic Processes (SS 2018)
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Problem Set 6

(discussion on June 14th)

Drift and subdiffusion to an absorbing boundary

At time $t_0 = 0$ a density $p(x, t_0) = \delta(x - x_0)$ is created at a position x_0. We consider a subdiffusive random process with a bias, evolving according to the fractional diffusion equation

$$\partial_t P = {}_0D_1^{1-\alpha} \left(-v_\alpha \partial_x P + K_\alpha \partial_x^2 P \right) . \tag{1}$$

(a) How is v_α defined in the diffusion limit of a CTRW?

(b) Find and solve the evolution equations for the mean and the variance of the density for natural boundary conditions.

(c) Solve the initial value problem in the presence of an absorbing boundary in the direction of the bias with the method of images. What are the exact probability current or its asymptotics ($t \to 0, t \to \infty$) at the absorbing boundary?

(d) Program the subdiffusive process as a CTRW with unit steps $\Delta x \in \{\pm h\}$ with $P_+ = q$ and $P_- = (1 - q)$, and asymptotically power-law distributed time increments $\Delta t \sim \tau^\alpha / t^{-(1+\alpha)}$. Generate $N \gg 1$ realizations and print the survival probability in a double logarithmic plot in the appropriate time and space increment limit.

(e) Compare your results with [H.Scher,E.W.Montroll, Phys.Rev.B 12(6), 2455 (1975)].

Lévy flights

Program an isotropic CTRW in two dimensions with time steps $\Delta t = \tau$ and asymptotically power-law distributed jump lengths $\Delta x \sim h^\alpha x^{-(1+\alpha)}$. Show example trajectories for different values of α and different length scales demonstrating the fractal nature of the trajectories visually.

Hints : The ratio $Z = X/Y$ of two uniformly distributed random variables $X, Y \sim U\,(0, 1]$ is Cauchy distributed. The power $Z = X^{-1/\alpha}$ of a uniformly distributed random variable $X \sim U\,(0, 1]$ is power-law distributed as $1 \leq Z \sim z^{-(1+\alpha)}$.