Pólya return problem

$$F_n(\underline{r}) = \Pr\left\{\begin{array}{l}\text{visit site } \underline{r} \text{ for the } 1^{st} \\ \text{time after } n \text{ steps}\end{array}\right\}$$

$$P_n(\underline{r}) = \Pr\left\{\begin{array}{l}\text{be at site } \underline{r} \\ \text{after } n \text{ steps}\end{array}\right\}$$

$$P_n(\underline{r}) = \underbrace{\delta_{n,0}\,\delta_{\underline{r},0}}_{\substack{\text{start from 0} \\ \text{at step 0} \\ \text{(initial condition)}}} + \sum_{k=1}^{n} F_k(\underline{r})\, P_{n-k}(0)$$

visiting $\underline{r}$ after k steps

return to $\underline{r}$ after $n-k$ steps

$$\Rightarrow \underbrace{\sum_n z^n P_n(\underline{r})}_{\equiv P(\underline{r},z) \text{ "z-transform"}} = \delta_{\underline{r},0}\, z^0 + \sum_n z^n \sum_{k=1}^{n} F_k(\underline{r})\, P_{n-k}(0)$$

$$F(\underline{r},z) \equiv \sum_n F_n(\underline{r})\, z^n$$

$$\Rightarrow \quad P(\underline{r},z) = \delta_{\underline{r},0} + F(\underline{r},z)\, P(0,z) \qquad \text{via convolution theorem}$$

$$\Rightarrow \quad F(\underline{r},z) = \frac{P(\underline{r},z) - \delta_{\underline{r},0}}{P(0,z)} \quad \Rightarrow F_0(0) = \lim_{z\to 0} F(0,z) = 0$$

$$F_n(0) = \Pr\left\{\begin{array}{l}\text{first return to } 0 \\ \text{after } n\geq 1 \text{ steps}\end{array}\right\}$$

Overall (cumulative) Pr to return to 0:

$$F(0) = \sum_{n=0}^{\infty} F_n(0).$$

$F(0) < 1$ random walker does not necessarily return to origin ($1 - F(0)$ is the escape probability): <u>transient walk</u>

$F(0) = 1$ <u>transient walk</u>: for $n \to \infty$ the walker revisits ∞ times

With $F(\underline{r}, z) = \sum_n F_n(\underline{r}) z^n \Rightarrow F(0) = \sum_n F_n(0) z^n \Big|_{z=1} = F(0,1)$

$$\Rightarrow F(0,z) = 1 - \frac{1}{P(0,z)}$$

$$\Rightarrow \underline{\underline{F(0) = 1 - \frac{1}{P(0,1)}}}$$

Consider now a random walk with probability p for a unit step to the right & q to the left. Consider the expression $pe^{i\vartheta} + qe^{-i\vartheta}$

$$(pe^{i\vartheta} + qe^{-i\vartheta})^2 = \underbrace{p^2 e^{2i\vartheta}}_{\text{2 steps to right}} + \underbrace{2pq}_{\substack{\text{no displacement} \\ \text{after 2 steps}}} + \underbrace{q^2 e^{-2i\vartheta}}_{\text{2 steps to left}}$$

$\Rightarrow$ generalising to any n $\Rightarrow$ coefficient of term $e^{ji\vartheta}$ of polynomial $(pe^{i\vartheta} + qe^{-i\vartheta})^n$ is the Pr to arrive @ site j after n steps

As $$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i\vartheta n} \, d\vartheta = \delta_{n,0}$$

$$\Rightarrow P_n(j) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (pe^{i\vartheta} + qe^{-i\vartheta})^n e^{-ji\vartheta}$$

$\Rightarrow \lambda(\vartheta) = pe^{i\vartheta} + qe^{-i\vartheta}$ is charact. function of the walker's displacement per step. We can write $\lambda(\vartheta) = \langle e^{i\vartheta x} \rangle$

We now consider $p = q = 1/2$.

$$\Rightarrow \lambda(\vartheta) = \frac{1}{2}\left(e^{i\vartheta} + e^{-i\vartheta}\right) = \cos\vartheta$$

$$\Rightarrow P_n(j) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos^n\vartheta \, e^{-ji\vartheta} \, d\vartheta = \frac{1+(-1)^{n+j}}{2^{n+1}} \binom{n}{\frac{n+j}{2}}$$

$$= \frac{1+(-1)^{n+j}}{2^{n+1}} \frac{n!}{\left(\frac{n+j}{2}\right)!\left(\frac{n-j}{2}\right)!}$$

This is our result from the first random walk derivation: n & j must have same parity & the displacement after n jumps never exceeds n.

Back to Pólya problem: $P_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \lambda^n(\vartheta) e^{i\vartheta x} d\vartheta$

$$P(0,z) = \sum_{n=0}^{\infty} P_n(0) z^n = \frac{1}{2\pi}\sum_{n=0}^{\infty}\int_{-\pi}^{\pi} \lambda^n(\vartheta) z^n \, d\vartheta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{n=0}^{\infty} \lambda^n(\vartheta) z^n \, d\vartheta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{d\vartheta}{1-z\lambda(\vartheta)}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{d\vartheta}{1-z\cos\vartheta} = \frac{1}{\sqrt{1-z^2}}$$

$\Rightarrow F(0,z) = 1 - \sqrt{1-z^2}$ and $\underline{\underline{F(0) = F(0,1) = 1}}$ in $d=1$ recurrent $\square$

<u>Higher dimensions:</u>

$P_n(0) = \left(\frac{1}{2\pi}\right)^d \int_{\Omega} \lambda^n(\underline{\vartheta}) \, d\underline{\vartheta}$: Ω over an elementary cell

Along the same steps:

$$P(0,z) = \left(\frac{1}{2\pi}\right)^d \int_{\Omega} \frac{d\underline{\vartheta}}{1-z\lambda(\underline{\vartheta})}$$

<u>Result:</u> $d=2$ $F(0) = 1$

$d=3$ $F(0) = 0.3405$ on cubic lattice

$\Rightarrow$ In $d=1$ the return is certain, in $d=3$ revisits to the same point are significantly reduced.

We say that in a search process one-dimensional walks lead to <u>oversampling</u>.