Trick to calculate: start with diffusion equation

$$x^2 \frac{\partial P}{\partial t} = \frac{\partial}{\partial t} x^2 P = D x^2 \frac{\partial^2 P}{\partial x^2} \quad \Big| \int_{-\infty}^{\infty} dx \,.$$

$$\frac{d}{dt} \langle x^2 \rangle = D \left\{ \left[x^2 P' \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} 2x \frac{\partial P}{\partial x} dx \right\} = D \left\{ -\left[2xP \right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} 2P \, dx \right\} = 2D$$

$\Rightarrow \langle x^2 \rangle = 2Dt.$

Fick's derivation of the diffusion equation:

(1) Continuity equation: Given a probability density $P(\underline{r}, t)$ with $\iiint_{-\infty}^{\infty} P(\underline{r}, t)\, dV = 1$, and the probability flux $\underline{S}(\underline{r}, t)$:

$$\oint\!\!\oint_{\delta V} \underline{S}(\underline{r}, t)\, d\underline{A} = -\frac{d}{dt} \iiint_V P(\underline{r}, t)\, dV = -\frac{d}{dt} \mathcal{S}(t)$$

where $\mathcal{S}(t)$ is the survival probability.
With the divergence theorem (Gauss' theorem):

$$\oint\!\!\oint_{\delta V} \underline{S}(\underline{r}, t)\, d\underline{A} = \iiint_V \nabla \cdot \underline{S}(\underline{r}, t)\, dV$$

we obtain the integral form of the continuity equation:

$$\iiint_V \nabla \cdot \underline{S}(\underline{r}, t)\, dV = -\frac{d}{dt} \iiint_V P(\underline{r}, t)\, dV = -\iiint_V \frac{\partial P(\underline{r}, t)}{\partial t}\, dV$$

This relation is valid $\forall\ (V, \delta V) \Rightarrow$ differential form:

$$\frac{\partial}{\partial t} P(\underline{r}, t) = -\nabla \cdot \underline{S}(\underline{r}, t) = -\operatorname{div} \underline{S}(\underline{r}, t).$$ continuity equation

(2) Fick's first law:

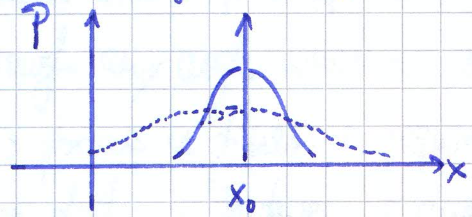
$$\underline{S} = -D \nabla P(\underline{r}, t)$$ the flux is proportional to the gradient of the probability density function P.

$$\Rightarrow \frac{\partial}{\partial t} P(\underline{r},t) = -\nabla \cdot \underline{S}(\underline{r},t) = D\nabla^2 P(\underline{r},t) \quad \text{Fick's second law}$$

In the following, for simplicity we will deal with the one-dimensional case.

Boundary value problems:

(1) Reflecting boundary at $x=0$ & initial condition $P_0(x) = \delta(x-x_0)$, $x_0 > 0$



$$\left.\frac{\partial P(x,t)}{\partial x}\right|_{x=0} = 0$$

Solution: (a) Standard method: Laplace transform & solution of ODE in x.

(b) Method of images: The portion of the probability "leaking" out to $x<0$ of the Gaussian $(4\pi D t)^{-1/2} \exp(-[x-x_0]^2/(4Dt))$ is compensated by the influx of probability from a mirror source at $-x_0$. The resulting Green's function solves the boundary value problem:

$$Q(x,t) = G(x-x_0,t) + G(x+x_0,t) \quad \text{where } G(x,t) = \frac{1}{\sqrt{4\pi D t}} e^{-x^2/4Dt}$$

Moreover Q is normalized. Thus the image Q is the sought-after solution.

Diffusion on average moves the particle away from the origin:

$$\langle x\rangle = \int_0^\infty Q(x,t)\,dx = 2\sqrt{\frac{Dt}{\pi}}\, e^{-x_0^2/4Dt} + x_0 \,\mathrm{erf}\left(\frac{x_0}{\sqrt{4Dt}}\right) \sim \sqrt{\frac{4Dt}{\pi}}.$$

(2) Absorbing boundary conditions at $x=0$:

$P(0,t) = 0$

Image solution $Q(x,t) = G(x-x_0,t) - G(x+x_0,t)$.

Survival probability: $\mathcal{S}(t) = \int_0^\infty Q(x,t)\,dx = \mathrm{erf}\left(\frac{x_0}{\sqrt{4Dt}}\right) \sim \frac{x_0}{\sqrt{\pi D t}}$.

III. RANDOM WALKS.

1905 Karl Pearson poses the problem of the random walk, being interested in the dynamics how mosquito populations invade cleared jungle regions:
"Can any of your readers refer me to a work wherein I should find the solution to the following problem [...]:
A man starts from a point O and walks l yards in a straight line; he then turns through any angle whatever and walks l yards in a second straight line. He repeats this process n times. I require the probability that after these n stretches he is at a distance between r and $r+dr$ from his starting point, O."

Lord Rayleigh pointed out one week later that in the limit of large n the solution is a Gaussian.

Consider a jump process on a one-dimensional lattice with spacing a. At each step the probability to jump left or right is equal.
$\rightsquigarrow$ Probability to reach site m after N steps ($N \pm m$ are always even!)

$$g(m,N) = \left(\frac{1}{2}\right)^N \frac{N!}{\underbrace{\left[\frac{1}{2}(N-m)\right]!}_{\frac{N-m}{2}\text{ jumps to left}}\underbrace{\left[\frac{1}{2}(N+m)\right]!}_{\frac{N+m}{2}\text{ jumps to right}}}$$

Stirling's formula: $\log N! = \left(N+\frac{1}{2}\right)\log N - N + \frac{1}{2}\log 2\pi + \mathcal{O}(N^{-1})$

Expansion of log: $\log(1+z) \sim z - \frac{z^2}{2}$, $z \ll 1$.

$$\Rightarrow \quad \log g(m,N) \approx -\frac{1}{2}\log N + \log 2 - \frac{1}{2}\log 2\pi - \frac{m^2}{2N}$$

$$g(m,N) \simeq \sqrt{\frac{2}{\pi N}}\,\exp\left(-\frac{m^2}{2N}\right)$$

m is either always even (N even) or always odd (N odd)

$\Rightarrow$ probability to meet the walker at site m:

$$P(m,N) = \frac{1+(-1)^{N-m}}{2}\, g(m,N) \approx \frac{1+(-1)^{N-m}}{\sqrt{2\pi N}} \exp\left(-\frac{m^2}{2N}\right). \quad (*)$$

Note that $P(m,N)=0$ for $|m|>N$ for exact expression, but not for approximation. This result is one form of the central limit theorem (CLT): the (normalised) sum of independent random variables with finite variance is well approximated by a random variable with a Gaussian distribution in the limit of large numbers.

Even for $N=10$ the approximation $(*)$ containing the Gaussian is almost exact (normalisation is 0.999594).

The factor $\frac{1+(-1)^{N-m}}{2}$ is essential for the normalisation.

Continuum limit:

Position of the walker is $x = ma$:

$$P(x,N) \approx \sqrt{\frac{1}{2\pi N a^2}} \exp\left(-\frac{x^2}{2Na^2}\right)$$

from Jacobian

Time $t = N\Delta t$ and diffusion constant $K = \frac{a^2}{2\Delta t}$

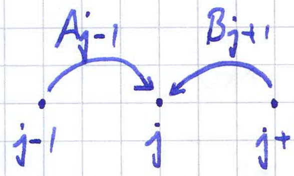
$$\Rightarrow P(x,t) \approx \sqrt{\frac{1}{4\pi K t}} \exp\left(-\frac{x^2}{4Kt}\right).$$

IV. THE FOKKER-PLANCK EQUATION.

Let us start from a random walk, but now the probability to jump left or right depends on the walker's position: we start from the discrete master equation

$$W_j(t+\Delta t) = A_{j-1} W_{j-1}(t) + B_{j+1} W_{j+1}(t)$$

Pr to find walker at j at $t+\Delta t$



$A_j + B_j = 1$ jump w/ Pr $=1$.

Taylor expansions: $W_j(t+\Delta t) \approx W_j(t) + \Delta t \frac{\partial}{\partial t} W_j(t)$

$$A_{j-1} W_{j-1} \approx A(x) W(x) - \Delta x \frac{\partial}{\partial x} A(x) W(x) + \frac{(\Delta x)^2}{2} \frac{\partial^2}{\partial x^2} A(x) W(x)$$

$$B_{j+1} W_{j+1} \approx B(x) W(x) - \Delta x \frac{\partial}{\partial x} B(x) W(x) + \frac{(\Delta x)^2}{2} \frac{\partial^2}{\partial x^2} B(x) W(x)$$

$$\Rightarrow W(x,t) + \Delta t \frac{\partial W}{\partial t} = \underbrace{\left[A(x) + B(x)\right]}_{\equiv 1} W(x,t) + \Delta x \frac{\partial}{\partial x} \left(B(x) - A(x)\right) W(x,t)$$
$$+ \frac{(\Delta x)^2}{2} \frac{\partial^2}{\partial x^2} \left[A(x) + B(x)\right] W(x,t)$$

$$\frac{\partial}{\partial t} W(x,t) = \frac{\Delta x}{\Delta t} \frac{\partial}{\partial x} \left[B(x) - A(x)\right] W(x,t) + \frac{(\Delta x)^2}{2\Delta t} \frac{\partial^2}{\partial x^2} W(x,t)$$

Continuum limit $\Delta x \to 0$, $\Delta t \to 0$:

$$\frac{\partial}{\partial t} P(x,t) = \frac{\partial}{\partial x} \frac{V'(x)}{m\eta} P(x,t) + K \frac{\partial^2}{\partial x^2} P(x,t)$$ Fokker-Planck equation

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$$\lim_{\Delta x \to 0, \Delta t \to 0} \frac{\Delta x}{\Delta t} \left[B(x) - A(x)\right] = \frac{V'(x)}{m\eta}$$

$$\lim_{\Delta x \to 0, \Delta t \to 0} \frac{(\Delta x)^2}{2\Delta t} = K$$

for proper limit we must have that $B(x) - A(x) \approx \Delta x$

$F(x) = -V'(x)$ is the external force. η is the friction coefficient.

In a confining potential the long-time limit of $P(x,t)$ reaches the equilibrium distribution:

$$P_{eq}(x) = \mathcal{N} \exp\left(-\frac{V(x)}{k_B T}\right).$$

$$\frac{\partial}{\partial t} P(x,t) = 0 \Rightarrow 0 = \left(\frac{\partial}{\partial x} \frac{V'(x)}{m\eta} + K \frac{\partial^2}{\partial x^2}\right) P(x,t)$$

$$\rightsquigarrow \quad 0 = \left(\frac{V'(x)}{m\eta} + K \frac{\partial}{\partial x}\right) P_{eq}(x)$$

$$\Rightarrow \quad P_{eq}(x) = \mathcal{N} \exp\left(-\frac{V'(x)}{K m\eta}\right)$$

$$\Rightarrow \quad \underline{\underline{K = \frac{k_B T}{m\eta}}}$$ Einstein - Stokes - Smoluchowski relation (fluctuation - dissipation relation).

<u>Constant external force F_0:</u>

$$\frac{\partial}{\partial t} P(x,t) = -\frac{F_0}{m\eta} \frac{\partial}{\partial x} P(x,t) + K \frac{\partial^2}{\partial x^2} P(x,t)$$ sometimes called diffusion-advection eq.

(i.) <u>Similarity solution</u> with Galilei (wave) variable $\xi = x - \frac{F_0}{m\eta} t$

$$P(x,t) = G\left(x - \frac{F_0}{m\eta} t, t\right) \therefore G(x,t) = \frac{1}{\sqrt{4\pi K t}} \exp\left(-\frac{x^2}{4Kt}\right).$$

(ii.) <u>Second Einstein relation (linear response):</u>

$$\frac{d}{dt}\langle x(t)\rangle = \frac{F_0}{m\eta} \quad \Rightarrow \langle x(t)\rangle = \frac{F_0}{m\eta} t = \frac{F_0 K}{k_B T} t$$

w/o force we know that: $\langle x^2(t)\rangle_0 = 2Kt$

$$\Rightarrow \underline{\langle x(t)\rangle_{F_0} = \frac{F_0}{2 k_B T} \langle x^2(t)\rangle_0}.$$

V. The continuous time random walk process

EW Montroll, GH Weiss, J Math Phys 6, 167 (1965).
H Scher, M Lax, Phys Rev B 7, 4502 (1973).
MF Shlesinger, J Stat Phys 10, 421 (1974).
H Scher, EW Montroll, Phys Rev B 12, 2455 (1975).
J Klafter, R Silbey, Phys Rev Lett 44, 55 (1980).

Basic ingredient: Waiting time distribution $\psi(t)$
After each jump the walker waits for a random time drawn from the probability density function $\psi(t)$. We use $\psi(u) = \langle e^{-ut} \rangle$

Sticking probability for not moving:

$$\phi(t) = \int_t^\infty \psi(t')\,dt' = 1 - \int_0^t \psi(t')\,dt' \Rightarrow \phi(u) = \frac{1}{u} - \frac{\psi(u)}{u} = \frac{1-\psi(u)}{u}$$

PDF of occurrence of ith step at time $t = t_1 + t_2 + \ldots + t_i$

$$\mathcal{L}\{\psi_i(t)\} = \underline{\underline{\psi_i(u) =}} \langle e^{-ut} \rangle = \langle e^{-u(t_1+t_2+\ldots+t_i)} \rangle \underset{\text{independent}}{\underline{\underline{t_i}}}$$

$$= \langle e^{-ut_1} \rangle \langle e^{-ut_2} \rangle \cdots \langle e^{-ut_i} \rangle = \langle e^{-ut} \rangle^i = \underline{\underline{\psi^i(u)}}$$

$\Rightarrow$ Probability that the walker has jumped exactly i times up to time t:

$$Q_i(t) = \int_0^t \psi_i(t')\,\phi(t-t')\,dt'$$

$$\rightsquigarrow Q_i(u) = \psi_i(u)\,\phi(u) = \frac{1-\psi(u)}{u}\,\psi^i(u).$$

$\Rightarrow$ Probability to find walker at site n at time t:

$$P(n,t) = \sum_{i=0}^\infty p_i(n)\,Q_i(t) \quad \therefore \quad p_i(n) \text{ is probability at site } n \text{ after } i\text{th step.}$$

$$\rightsquigarrow P(n,u) = \frac{1-\psi(u)}{u} \sum_{i=0}^\infty p_i(n)\,\psi^i(u) = \frac{1-\psi(u)}{u}\,p_0(n) + \frac{1-\psi(u)}{u} \sum_{i=1}^\infty p_i(n)\,\psi^i(u)$$

(**)

Continuum limit:

$p_0(x) = \delta(x)$ initial condition

$p_{i+1}(x) = p_i(x) + \frac{a^2}{2}\frac{\partial^2}{\partial x^2} p_i(x) + \dots$ according to our results from chapter IV.

$$\Rightarrow \mathcal{P}(x,u) = \frac{1-\psi(u)}{u}\delta(x) + \frac{1-\psi(u)}{u}\sum_{i=1}^{\infty}\left(p_{i-1}(x) + \frac{a^2}{2}\frac{\partial^2}{\partial x^2} p_{i-1}(x)\right)\psi^i(u)$$

but from (**) $\frac{1-\psi(u)}{u}\sum_{i=1}^{\infty} p_{i-1}(x)\psi^i(u) = \frac{1-\psi(u)}{u}\sum_{i=0}^{\infty} p_i(x)\psi^{i+1}(u)$

$$= \mathcal{P}(x,u)\,\psi(u)$$

$$\Rightarrow \mathcal{P}(x,u) = \frac{1-\psi(u)}{u}\delta(x) + \psi(u)\left(\mathcal{P}(x,u) + \frac{a^2}{2}\frac{\partial^2}{\partial x^2}\mathcal{P}(x,u)\right)$$

$$\underline{\underline{\mathcal{P}(x,u) - \frac{\delta(x)}{u} = \frac{\psi(u)}{1-\psi(u)}\frac{a^2}{2}\frac{\partial^2}{\partial x^2}\mathcal{P}(x,u)}}$$

CTRW with arbitrary $\psi(t)$ in the continuous space approximation.

(1.) Back to the diffusion equation:

Sharp waiting time PDF: $\psi(t) = \delta(t-\tau) \Rightarrow \psi(u) = e^{-u\tau} \approx 1 - u\tau + \dots$

many step ≡ long time corresponds to short u ⇝ powers in $(u\tau)^2 \dots$ neglected

Same result for Poissonian $\psi(t)$:

$$\psi(t) = \tau^{-1} e^{-t/\tau} \Rightarrow \psi(u) = \frac{1}{u+\tau^{-1}} \approx 1 - u\tau + \dots$$

In fact, this behaviour is universal, as long as the characteristic waiting time ~~...~~ remains finite:

$$\langle\tau\rangle = \int_0^{\infty}\psi(t)\,t = \int_0^{\infty} t\,\psi(t)\,e^{-ut}dt\Big|_{u=0} = -\frac{d\psi(u)}{du}\Big|_{u=0}$$

$\Rightarrow$ for finite $\langle\tau\rangle$ we have $\psi(u) \doteq 1 - \langle\tau\rangle u + \dots$

Up to first order in τ: $\dfrac{\psi(u)}{1-\psi(u)} = \dfrac{1-u\tau}{u\tau} = \dfrac{1}{u\tau} - 1$

$$\Rightarrow P(x,u) - \frac{\delta(x)}{u} = \left(\frac{1}{u\tau} - 1\right) \frac{a^2}{2} \frac{\partial^2}{\partial x^2} P(x,u)$$

The diffusion constant is $K = \lim\limits_{a\to 0,\, \tau \to 0} \dfrac{a^2}{2\tau}$

$\Rightarrow$ the term $\lim\limits_{a\to 0,\, \tau\to 0} a^2 \longrightarrow 0$

$$\Rightarrow P(x,u) - \frac{\delta(x)}{u} = \frac{1}{u} K \frac{\partial^2}{\partial x^2} P(x,u)$$

After inverse Laplace transformation:

$$P(x,t) - \delta(x) = \int_0^t K \frac{\partial^2}{\partial x^2} P(x,t')\, dt' \quad \text{integral form of diffusion eq.}$$

$$\rightsquigarrow \frac{\partial}{\partial t} P(x,t) = K \frac{\partial^2}{\partial x^2} P(x,t) \quad \checkmark$$

(2) Diverging characteristic waiting time:

Assume the form $\psi(u) = e^{-(u\tau)^\alpha}$, $0<\alpha<1$ for the waiting time PDF. This is actually the characteristic function of a one-sided Lévy stable law, resulting from a generalisation of the CLT for random variables with diverging variance. The leading order of the Laplace inversion is $\psi(t) \sim \tau^\alpha / t^{1+\alpha}$.

In particular, we see that $\langle \tau \rangle = \int_0^\infty t\, \psi(t)\, dt = -\dfrac{d\psi(u)}{du}\Big|_{u=0} = \tau^\alpha u^{\alpha-1} e^{-(u\tau)^\alpha}\Big|_0 \to \infty$.

This process is scale-free, i.e. there is no time scale separating single jump events from the limit of a large number of jumps $\rightsquigarrow$ ageing, non-ergodicity.

In Laplace space, we find

$$P(x,u) - \frac{\delta(x)}{u} = \frac{1}{u^\alpha} K_\alpha \frac{\partial^2}{\partial x^2} P(x,t) \;\therefore\; K_\alpha = \lim_{a\to 0,\, \tau\to 0} \frac{a^2}{2\tau^\alpha} \;\therefore\; [K_\alpha] = \frac{cm^2}{sec^\alpha}. \quad (*)$$

Mean squared displacement: Integrate over $\int x^2 dx$:

$$\langle x^2(u)\rangle = \frac{2K_\alpha}{u} \quad \Rightarrow \quad \underline{\underline{\langle x^2(t)\rangle = \frac{2K_\alpha}{\Gamma(1+\alpha)} t^{\alpha}}}$$

Processes with $\langle x^2\rangle \sim t^\alpha$, $\alpha \neq 0$ are called anomalous diffusion.

$0<\alpha<1$: subdiffusion

There also exists superdiffusion with $\alpha > 1$ ($\alpha = 2$: ballistic, wave-like)

Relaxation of single Fourier modes:

Fourier transform of eq ($*$):

$$P(k,u) - \frac{1}{u} = -K_\alpha \frac{k^2}{u^\alpha} P(k,u)$$

$$\leadsto P(k,u) = \frac{1}{u + K_\alpha k^2 u^{-\alpha+1}} = \frac{1}{u + u^{1-\alpha}\tau_*^{-\alpha}} \equiv \phi(u)$$

$\tau_*^{-\alpha} \equiv K_\alpha k^2$ is a time scale @ k fixed.

This Laplace image defines the Mittag-Leffler function:

$$\phi(t) = E_\alpha\left(-[t/\tau_*]^\alpha\right) = \sum_{n=0}^{\infty} \frac{\left(-[t/\tau_*]^\alpha\right)^n}{\Gamma(1+\alpha n)} \sim 1 - \frac{(t/\tau_*)^\alpha}{\Gamma(1+\alpha)}$$

In the limit $\alpha = 1$ we recover the standard exponential function

For $0<\alpha<1$ the Mittag-Leffler function has the asymptotic expansion:

$$\phi(t) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(\tau_*/t)^{\alpha n}}{\Gamma(1-\alpha n)} \sim \frac{(\tau_*/t)^\alpha}{\Gamma(1-\alpha)}$$

The Mittag-Leffler function interpolates between an initial stretched exponential law $\phi \sim \exp\left(-\frac{(t/\tau_*)^\alpha}{\Gamma(1+\alpha)}\right)$ and a terminal inverse power-law.

For $\alpha = 1/2$, the Mittag-Leffler function reduces to:

$$E_{1/2}\left(-[t/\tau_*]^{1/2}\right) = e^{t/\tau_*} \operatorname{erfc}\left(\left[\frac{t}{\tau_*}\right]^{1/2}\right).$$

An alternative derivation of CTRW with a jump length distribution $\lambda(x)$.

Jump length distribution $\lambda(x)$ can have finite or infinite variance
$\langle\delta x^2\rangle = \int_{-\infty}^{\infty} x^2 \lambda(x)\,dx$.

Consider the PDF of just having arrived at x at time t, $\eta(x,t)$. This PDF fulfills the generalised master equation:

$$\eta(x,t) = \int_{-\infty}^{\infty} dx' \int_0^t dt'\, \eta(x',t')\, \lambda(x-x')\, \psi(t-t') + \underbrace{\delta(x)\,\delta(t)}_{\text{initial condition needed @ } t=0}$$

$\eta(x,t)$ is related to the position PDF $P(x,t)$ via

$$P(x,t) = \int_0^t \eta(x,t')\,\phi(t-t')\,dt'$$ of having arrived at t' and not having moved since.

$$\Rightarrow P(k,u) = \eta(k,u)\,\phi(u) = \frac{1-\psi(u)}{u}\left\{\frac{1}{1-\lambda(k)\,\psi(u)}\right\}$$

$$\underline{\underline{P(k,u) = \frac{1-\psi(u)}{u}\,\frac{1}{1-\lambda(k)\,\psi(u)}}}$$

(1.) Gaussian jump length distribution:

$$\lambda(k) = \exp(-\tfrac{1}{2}\sigma^2 k^2) \sim 1-\tfrac{1}{2}\sigma^2 k^2 \quad ; \text{ with } \psi(u) \sim 1-u\tau$$

$$\Rightarrow 1-\lambda(k)\psi(u) \sim 1-\left(1-\tfrac{\sigma^2}{2}k^2\right)(1-u\tau) \sim u\tau + \frac{\sigma^2}{2}k^2$$

$$\rightsquigarrow P(k,u) = \frac{\tau}{u\tau+\frac{\sigma^2}{2}k^2} = \frac{1}{u+K_\alpha k^2} \;:\; K = \lim_{\sigma\to0,\tau\to0}\frac{\sigma^2}{2\tau}$$

$$\rightsquigarrow P(x,t) = \frac{1}{\sqrt{4\pi K t}}\exp\left(-\frac{x^2}{4Kt}\right).$$

(2.) α-stable jump length distribution:

$$\lambda(k) = \exp\left(-\frac{\sigma^\mu}{\mu}|k|^\mu\right) \sim 1-\frac{\sigma^\mu}{\mu}|k|^\mu \quad \rightsquigarrow \langle\delta x^2\rangle \to \infty\,,\quad \lambda(x) \sim \frac{\sigma^\mu}{|x|^{1+\mu}}$$

$$P(k,u) = \frac{1}{u + K^\mu |k|^\mu} \rightsquigarrow P(k,t) - \exp(-K^\mu |k|^\mu t)$$

$$\rightsquigarrow P(x,t) \sim \frac{K^\mu t}{|x|^{1+\mu}} \quad \text{"Lévy flight"}$$

Question: What is the dynamic equation for $P(x,t)$ when in its Fourier-Laplace transform terms such as u^α and $|k|^\mu$ occur?

$\rightsquigarrow$ Fractional calculus.

Fractional order differintegration:

1695 in a letter to GA de l'Hospital, GW Leibniz writes: Thus it follows that $d^{1/2}x$ will be equal to $x\sqrt{dx:x}$, an apparent paradox, from which one day useful consequence will be drawn."

(1.) Phenomenological introduction of fractional order differentiation:

$$\frac{d^m x^n}{dx^n} = \frac{n!}{(n-m)!} x^{n-m} \quad \text{for integers } m,n$$

generalisation through introduction of Γ function:

$$\frac{d^\mu x^\nu}{dx^\nu} \equiv \frac{\Gamma(\nu+1)}{\Gamma(\nu-\mu+1)} x^{\nu-\mu} \quad \text{for real-valued } \mu,\nu$$

NB: In this definition the derivative of a constant is not zero:

$$\frac{d^\mu x^0}{dx^\nu} = \frac{1}{\Gamma(1-\mu)} x^{-\mu}$$

Application to exponential function:

incomplete γ function

$$\frac{d^\mu}{dx^\mu} e^{-t} = \frac{d^\mu}{dt^\mu} \sum_{n=0}^{\infty} \frac{(-t)^n}{\Gamma(1+n)} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{n-\mu}}{\Gamma(n-\mu+1)} = t^{-\mu} e^{-t} \gamma^*(-\mu,-t)$$

As $\gamma^*(-n,y) = y^n \rightsquigarrow$ formula reduces to well-known differentiation of exp.