University of Potsdam Institute of Physics and Astronomy
Lecture Stochastic Processes (SS 2018)
Prof. R. Metzler

# Problem Set 4

(discussion on may 24th)

### Subordinated Diffusion

A subordinated diffusion process, i.e. the diffusion limit of a CTRW with diverging mean waiting time distribution, evolves according to the time fractional Fokker-Planck equation

$$\partial_t P(x,t) = {}_0D_t^{1-\alpha} L_{FP} P(x,t) \tag{1}$$

where $L_{FP}$ is the Fokker-Planck operator for a usual diffusion process, i.e.

$$L_{FP}P = \partial_x \left( \frac{V'(x)}{m\eta_\alpha} P \right) + K_\alpha \partial_x^2 P. \tag{2}$$

The operator ${}_0D_t^{1-\alpha}$ is the Riemann-Liouville fractional time derivative. Applying the Laplace transform to a solution of Eq.(1) with initial distribution $P(x,0)$ we have

$$uP(x,u) - P(x,0) = u^{1-\alpha} L_{FP} P(x,u) \tag{3}$$

as shown in the lecture for $V(x) = const.$

(a) Show, using the series expansion of the Mittag-Leffler function

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(1+\alpha n)} \tag{4}$$

that

$$\int_0^\infty E_\alpha(-\lambda t^\alpha) e^{-ut} dt = \frac{1}{u + u^{1-\alpha}\lambda} \tag{5}$$

(b) Solve Eq.(1) for the time part of the separation ansatz for a given eigenvalue $-\lambda$ of the Fokker-Planck operator $L_{FP}$.

(c) Given a diffusion process $x = x(s)$ with a distribution $P(x,s)$ that solves the non-fractional diffusion equation, the process $y(t) = x(s(t))$ is called a subordinated diffusion process for a random process $s = s(t)$. Which time subordination processes are described by the time fractional FPE (1)?